TO THE INFINITE AND BACK AGAIN

Part I

Henrike Holdrege

A Workbook in Projective Geometry
To the Infinite and Back Again

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Part I

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Introduction

My intent in this book is to allow you, the reader, to engage with the fundamental concepts of projective geometry. I want to lead you into the experience of geometric thinking and thinking dynamically in transformations. The book is centrally concerned with practice that builds capacities. It is a pathway that brings clarity into thinking, challenges the imagination, and leads to surprising insights. On this journey you can begin to experience how concepts grow in an organic way and form a coherent and consistent whole. There is great beauty and depth in the ideas of projective geometry.

Each chapter of the book contains numerous exercises: exercises for living wakefully in thought; exercises that are enjoyable and foster inner quiet and concentration; exercises that strengthen and enhance trust in thinking. The exercises are not necessarily easy and may require time, effort, and, above all, perseverance. I believe that such study and inner work can be of real benefit.

Written as a workbook for self-study, this book is an introduction to projective geometry. It is intended for the lay-person with little or no knowledge and practice in geometry, and as a resource for high school and college math teachers. I have taught projective geometry to high school students and, for many years now, to adults. The book builds on my teaching experiences.

The book introduces key concepts and major theorems of projective geometry. Unfortunately, geometry is often taught in a strictly formalistic way, and books on projective geometry often begin with a set of axioms that simply must be accepted. But formalism is an end stage in mathematics. Before a field of mathematics can be formalized, there is the process of discovery, of wrestling with riddles, questions, and new ideas, and of finding and articulating new concepts, principles, and theorems. Mathematicians wrestled with the ideas of the infinitely distant for centuries, and along the way the living activity of “mathematizing” or “geometricizing” became crystalized in a more formal language.

When we only learn mathematics in a formalized way by applying rules, we may lose the living quality that is inherent in mathematics. We also run the danger of losing our feeling for whether we actually understand something or not. We don’t experience what it means to understand deeply and with growing interest. Therefore, we cannot harvest the fruits that come from the effort of studying and learning. Many people who got lost in math classes were probably lost because of the formalism, because they were not met in their own learning style, or because there was nothing for them to experience with which they could connect.

In ancient Greece, Euclid (who died around 270 BCE) took on the task of assembling and putting into a system the geometrical knowledge that had been articulated before him—think of the theorem of Pythagoras or the theorem of Thales. Euclid identified and articulated the axioms for Euclidean geometry, and provided proofs for all existing geometric theorems by logical inference from the axioms. This is the geometry that most of us learned in school.

One of the Euclidean axioms that you may be familiar with states: “Any two lines in a plane that are not parallel have a point in common.” This is one of the axioms that projective geometry transcends. Through a variety of exercises and constructions we will conceive, as you will see, a concept that the Greeks could not think. We will work to learn to think this challenging and mind-opening concept
through all chapters of this book. With our thinking we will move out of the finite into the infinite and back again.

Projective geometry is a modern form of geometry. It was developed over several centuries, beginning with the French mathematician and engineer Gérard Desargues (1591 - 1661), and culminating in the 19th century in the work of, for instance, Jean-Victor Poncelet (1788 - 1867), Karl Georg Christian von Staudt (1798 - 1867), and Arthur Cayley (1821 - 1895). In the 20th century, we find the synthetic approach to projective geometry, for example, in the work of the mathematicians Louis Locher-Ernst (1906 - 1962) and George Adams Kaufmann (1894 - 1963).

The approach to projective geometry I develop in this book is the synthetic approach, in contrast to the analytical approach. The synthetic approach is independent from all measurement and calculations, and thus independent from numbers and algebra. It is, therefore, especially accessible to all those people who “dislike math.” This approach can open doors for those for whom the long lost joy in geometry can be found again and be rekindled: the joy in clarity of thought, in a deepening understanding, and in the beauty of ideas and of geometric transformations.

Of course, doing geometry requires the capacity of logical thinking. We learn to hold and develop a line of thought. This clarity is important. The progression of thought in this book is not, however, simply linear. Later insights support and illuminate earlier ones. In the realm of thought, concepts are not isolated from each other. Each concept relates to other concepts; they carry and shed light on one another. They can even metamorphose into new ideas.

So as much as we need to strengthen our ability to develop logical sequences of thought, we also need to learn to see the more encompassing connections and relationships between ideas. There is a wonderful scene in Goethe’s Faust (Part I) that speaks of the difference between linear logical reasoning and “weaving in thought.”

Mephistopheles—the devil—disguises himself as Professor Faust when a young student enters Faust’s study to seek advice. The student is disheartened with the university and wishes guidance. Here is what Mephistopheles says, mixing sarcasm and deep truths as he so often does in this tragedy:

My friend, I shall be pedagogic,
And say you ought to start with Logic.
For thus your mind is trained and braced,
In Spanish boots it will be laced,
That on the road of thought maybe
It henceforth creep more thoughtfully,
And does not crisscross here and there,
Will-o’-the-wisping through the air.
Days will be spent to let you know
That what you once did at one blow,
Like eating and drinking so easy and free,
Can only be done with One, Two, Three.
Yet the web of thought has no such creases
And is more like a weaver’s masterpieces:
One step, a thousand threads arise,
Hither and thither shoots each shuttle,
The threads flow on, unseen and subtle,
Each blow effects a thousand ties.
The philosopher comes with analysis
And proves it had to be like this:
The first was so, the second so,
And hence the third and fourth was so,
And were not the first and the second here,
Then the third and fourth could never appear.
That is what all the students believe,
But they have never learned to weave.

Projective geometry presents the wonderful opportunity to weave in thought. I invite you to join me in the process.
Preparations

I drew most of the figures and illustrations in this book with graphite pencil and colored pencil in the hope of encouraging you, the reader, to do the same. Only when you draw the constructions yourself, will you enter fully the inner process.

Many drawings that you will be asked to execute in this book require accuracy. If you have little practice in geometric drawing—as most adults do—the following hints might be of help.

You will need the right tools: a straight edge (ruler), a set triangle, an eraser, a pencil sharpener, a hard graphite pencil (H or HB), a variety of good quality colored pencils, paper, and maybe a smooth board to place the paper on. A compass, in this workbook, is only needed as an auxiliary tool; in the second volume it will be needed for drawing circles.

When drawing a line, remember that, ideally, the geometric line has no width. The pencil line needs to be thin, and, if passing through a point, it needs to pass through the point and not miss it. Use a well sharpened graphite pencil, not a colored pencil, to draw the line. If you wish to emphasize a line by color, draw afterwards a color line close to but not over the graphite line.

If several lines are to be drawn through the same point, prick a tiny hole (with the needle of the compass) in the place of the point. Insert the sharp pencil tip in this hole (you can feel it!) and position the ruler by letting it touch the pencil. Don’t draw several lines through the point, but lift the pencil up in the proximity of it (see my drawings in the first lesson of the first chapter), so that the point does not become too “expansive.” Remember, ideally, a point has no extension whatsoever!

To highlight a point, don’t draw a colorful blotch, rather, circle the point with graphite pencil or colored pencil.

You will sometimes need to draw a line that is parallel to a given line. Here is a fast and accurate method of drawing parallel lines (see the photos, from top to bottom):

You need two tools: a straight edge and a set triangle. Place one side of the set triangle neatly alongside the line that is given. Then place the straight edge next to one of the other two triangle sides. Hold the straight edge firmly in place, and slide the set triangle along it into the desired position. Draw the line. You will need some practice for this method to become easy for you.

In chapter 6, you will be asked to draw lines that are vertical (i.e. that are perpendicular to a horizontal line). Rather than
constructing such a line with the help of a compass, I recommend that you use the right angle of a set triangle. (The set triangle shown in the photos has a right angle.) Another tool that you can use for this task is a tool that will be needed in certain chapters of the second volume. It is a graph ruler.

Lastly, an important technique for achieving clear and beautiful drawings is the shading of an area with a colored pencil. It requires some practice. Shade with the colored pencil held at a slant. Don’t color the paper with the sharp tip of the colored pencil in order to avoid visible strokes.

**Designations in this book**

As is customary in geometry textbooks, I designate points with capital letters (point A, B, ... Y, Z), and lines with small letters (line a, b, ... y, z). For the line through points A and B, I write “line AB.” For the point of intersection of two lines AB and CD, I write “AB x CD.” If a triangle has the corners A, B, and C, I speak in the text about the “triangle ABC.”

**Technical terms**

Most terms that I will use in this book I will introduce in the context as they are needed. But in addition, on the following page, you will find a section containing basic terms—entitled “Ten Basic Entities”—that are much used in projective geometry and throughout this book. Think of page 6 as a reference page for the whole book.

**Make choices**

You will find certain exercises and lessons in this book to be more difficult than others. It is perfectly fine to skip those that you find too difficult for you at the time, and maybe return to them at a later time. I trust that you will find your own relationship to the exercises in this book—that there will be those that you prefer to do and others to which you give little attention.

This is especially true for teachers who wish to use this book as a teaching resource. As a teacher, you need to meet the needs of your students, and you are given only a certain amount of time for teaching the subject. You need to pick and choose from the material in this book. Moreover, Part II of *To the Infinite and Back Again* (which is forthcoming) will offer more on projective geometry, and much of it is as essential as the content of Part I.
Ten basic entities

Projective geometry works with **points**, **lines**, and **planes**. There are two activities that bring these elements into a relationship with each other, **connecting** and **intersecting**: Two points are connected by the line they have in common (the **connecting line**). Two planes intersect (meet) in a line (the **line of intersection**), three planes intersect in a point (the **point of intersection** or **meeting point**) etc.

Points, lines, and planes are part of each other: Points and lines lie in planes; planes and lines lie in points; points lie on lines; planes lie in lines. Here we distinguish ten basic entities in projective geometry:

We can view a **plane** as an entity unto itself. There is an infinitude of planes in three-dimensional space. When we view a plane as the carrier of all the points that lie in that plane, we speak of a **field of points**. When we view a plane as the carrier of all the lines that lie in that plane, we speak of a **field of lines**.

We can view a **point** as an entity unto itself. There is an infinitude of points in three-dimensional space. When we view a point as the carrier of all the planes in that point, we speak of a **bundle of planes**. When we view a point as the carrier of all the lines in that point, we speak of a **bundle of lines**.

We can view a **line** as an entity unto itself. There is an infinitude of lines in three-dimensional space. When we view a line as the carrier of all the points that lie on that line, we speak of a **range of points**. When we view a line as the carrier of all the planes in that line, we speak of a **sheaf of planes**.

Lastly, when we view a point in a particular plane as the carrier of all the lines in that point that are also in that particular plane, we speak of a **pencil of lines**.
Prelude

Form and Forming

Picture a triangle—to start with, picture an equilateral triangle (i.e. a triangle with all three sides of equal length). Next, picture one of the three sides getting longer. Observe which sides and angles change also and which do not. Next, let one angle get larger (or smaller). Again, observe the whole triangle in its changing and unchanging aspects. Continue to transform the imagined triangle willfully and wakefully. Let it take on all kinds of shapes. They can be acute, right, or obtuse triangles. Conclude by transforming the triangle back into an equilateral triangle.

Now, picture a circle. Let the circle gradually get larger without displacing its center. Let the circle become very large. Next, let the circle contract and get very small, but don’t let it disappear into the center. Let it expand and contract again several times. End with the mental picture of a well-formed circle of a comfortable size.

Each of these two picturing exercises may take several minutes. While they seem to be simple they can serve as concentration exercises and assist us in gaining focus and becoming centered.

Reflections on the exercises

In the triangle transformation exercise, we cannot change a part without changing the whole.

Every concrete triangle has a certain shape and size. It and all its parts are specific. By mentally transforming an imagined triangle we overcome the specificity. The pictured triangle becomes fluid, but is always governed by the formative principle inherent in all triangles.

When we draw a triangle, it must always be a specific one. Placing two triangles next to each other, we can ask, “How can I transform the one into the other?” In mental picturing, we can perform a fluid, continuous transformation from one shape into the other. On paper, we cannot do that. We have to draw separate “snapshots” in order to indicate a continuous transformation.

The idea, triangle, is inherent and expressed in every particular triangle. It in-forms it. While we can picture and draw particular triangles, and while we can picture a continuous triangle transformation, we cannot picture the idea itself. It remains invisible, so to speak, but is nevertheless at work in every imagined triangle and in every triangle we find in the world.

The idea of a triangle or circle can be articulated as a definition or a description. That way the idea remains abstract. When we perform the triangle transformation exercise, in contrast, the idea becomes palpable. We transform each triangle in accordance with the idea. In our inner activity we put to work, and also experience, the idea as a formative principle.

When we do the circle-picturing exercise after the triangle transformation exercise, we can experience the contrast between these two different form principles.
With these mental exercises we have entered the inner space of thought where all true mathematical activity takes place. It is not through looking into the outer world—for instance, not through measuring angle openings or lengths of line segments—that we arrive at insights that hold true for all triangles, or all circles. Drawings can support us in our work; we might need them. However, mathematical reasoning is a process of pure thought.

Since the elements of geometry, and of mathematics in general, are concepts not derived from sense experiences, mathematical truths cannot be investigated empirically. While truths about triangles cannot be found in the outside world, no triangle in the outside world will contradict these truths—as long as they are triangles.

One virtue, therefore, of engaging in mental picturing exercises is that we learn to be present in that thought space and to dwell in it with full consciousness, intention, and clarity. We learn to work and observe in that space. Picturing is an inner activity. It takes effort to willfully form and transform mental images of geometric forms.

It is possible that I picture concentrically growing or contracting circles or changing triangles as I have seen them in animations. In this case, the inner activity of creating and transforming the form is missing. It happens without me. I am a mere onlooker. What matters, however, is the inner involvement. Without it, there is no exercise.

Such inner work can help us practice the kind of active thinking we need in order to explore the formative principles at work in nature, in plants, in animals, as well as in social life. In this book we will make ample use of mental-picturing exercises focused on form transformations. Projective geometry offers a wealth of opportunities to practice active thinking.

Working with clay and with freehand drawing

For some people, forming, holding, and transforming mental images of a geometric form is not easy, or may even be impossible. In that case, there are other ways to facilitate a deepened experience of form and forming.

Forming a sphere with clay is one such way. We can form it from the inside out by putting small pieces of clay together and gradually forming a sphere. Or we can take a fist-sized clump of soft clay and form it into a sphere from the outside by gently pressing it into shape with outstretched palms of both hands.

Another very effective technique to assist in grasping geometric forms more fully and creatively is freehand drawing. Drawings of concentrically growing or contracting circles can be done after the freehand drawing of a circle has been practiced. A large piece of paper ("newsprint" quality is inexpensive and suits the purpose) is taped onto a table surface. You draw with crayons or fat, colored pencils while standing. First you take time to form an inner, mental image of a circle. Then you move your hand in a circling motion above the paper, shaping the form in the air before finally lightly tracing it on the paper.
When I introduce this exercise in a workshop and draw a freehand circle on the blackboard, I step back and take a look. What I have drawn is not a perfect circle. While we might not be able to do a better job, we all are able to perceive how the form on the board deviates from the ideal. We easily see where it is dented, bulging, lop-sided, or egg-shaped. This tells me that we all carry the ideal circle within us, and that the outward appearing form with its imperfections evokes in us the ideal form.

I once learned about a professional potter who was able to throw a perfect bowl or cup on her potter’s wheel at any time. It was a high skill and, for her, routine. If a cup, however, did turn out less than perfect, it was regarded as special and sold for more!

I sometimes begin a course with a group exercise. The room is set up for freehand drawing and the tables are placed in a loop. After preparing as described above, we each draw lightly a large freehand circle on the paper in front of us, to the best of our ability—just one single line. Then we put the crayon down and, on a signal that I give, we all move to the next station on our right. We pick up that station’s crayon, observe the form in front of us, acknowledge its imperfections, and draw a second form overlaying the first (always only a single line) by trying to improve it. When done, we put the crayon down and, together, move on to the next place. We continue until we return to the place where we started. Here, with more pressure on the crayon, we finalize the shape.

The outcome of the exercise is always reassuring. Each form has been worked on by every member of the group. (In the case of a big group, I form smaller groups of eight to twelve people.) We experience that collaboration is constructive and helpful. A critical eye that detects imperfections is asked for. We correct each other’s work. We sometimes experience that we can get caught in the existing form and are not able to change it forcefully enough. It takes confidence and trust to even out a bulge or lopsidedness. The forms in the end are pleasantly round and well-shaped, each a successful group effort.

Replacing our widespread competitive social habits as learners with helpful interactions and interest in each other’s work is an important key to a positive learning experience.
Johannes Vermeer (1632-1675)
The Music Lesson (1662-1664),
Royal Collection, Windsor Castle, England

Pieter de Hooch (1629 - 1684)
A Woman with a Child in a Pantry (1658),
Rijksmuseum, Amsterdam, Holland
Chapter 1

The Harmonic Net and the Harmonic Four Points

The elements projective geometry works with include points at infinity, lines at infinity, and the plane at infinity. In Euclidean geometry (after Euclid, Greek mathematician, 300 BCE), which is taught in middle and high schools, these concepts are not known. Euclidean geometry states, for instance, that two lines in a plane have a point in common unless they are parallel. For projective geometry, any two lines in a plane have a point in common, and for parallel lines this is a point at infinity.

That parallel lines should have a point in common is a strange and challenging idea, and for us, at first, it is not supported by any experience. In this chapter, I will attempt to provide a geometrical context and a thought experience by which to approach this idea. For this context I have chosen the harmonic net and the law of the harmonic four points.

I will give instructions for geometric drawings, and I strongly recommend that you, the reader, actually do the drawings. The figures and photos given in the text are meant as an aid for clarification and cannot be a substitute for your own work and engagement.

You will need paper (letter-size paper will be sufficient for most drawings), a compass, a straight edge (ruler), a well-sharpened pencil and a few colored pencils, eraser, pencil sharpener, and maybe a board to place the paper on. Most of all, you will need quiet time. I hope you enjoy the exercises and the discoveries you will make.

Course on projective geometry at The Nature Institute in spring 2015
Lesson 1: Drawing a harmonic net

With your paper placed horizontally, draw a line from edge to edge. The line should be parallel to the upper edge and about an inch away from it. Choose three points on this horizontal line, so that the point in the middle is equidistant from the other two points.

[To achieve this you can use the compass: Mark a point somewhere near the middle of the line. Open your compass and place the compass needle on the point you just marked. With the compass pencil, mark the points to the left and the right without changing the compass opening.]

Next, draw a line through each of the three points such that the three lines form a triangle below the horizontal line. Draw these and all following lines all the way to the edge of the paper. With a colored pencil, shade the triangle. (Your triangle need not resemble the one I have drawn.)

At this point you will have made all the free choices there are. From now on, every line that you will draw will be predetermined. As your drawing unfolds, step by step, line by line, you will make visible what in a sense is already there, albeit invisible.

To draw the next lines, take a look at your shaded triangle. Each of its corners is connected with two of the three points on the horizontal line, but not connected with one of the three. Draw for each corner the line that connects it with the point on the horizontal line it is not yet connected with.

You have now added three more lines to your drawing. They create new points of intersection which, again, are not connected with one of the three points on the horizontal line. Draw for every new intersection the third line.

Continue the process in this way, radiating out from the shaded triangle. Draw line after line. Proceed methodically; do not overlook a step.
You will soon find that several points are on the same line. However, they might seem to be not exactly on one line. Here is where you might misjudge the situation and make a mistake. It would be a mistake if you now drew several lines, more or less close to each other. Instead, you have to draw just one line with the points being more or less well met. It is our inability to draw with greater accuracy that causes this problem. It is not inherent in the construction itself. You will find these inaccuracies more pronounced toward the bottom and the sides of the paper, and less toward the horizontal line.

Continue drawing and enjoy the lawful unfolding of the net of lines. Toward the bottom and the sides of the paper you can complete the work; but upwards, toward the horizontal line, the process is never ending. You have to decide when to stop.

Looking at the drawing, you see that from each of the three points on the horizontal line there issues a set of lines. Together they form a pattern of triangles of which the initial shaded triangle is one. There are no gaps within this pattern, and each triangle is bordered by three other triangles.

We will now organize the drawing into a net of quadrilaterals. A quadrilateral is a four-sided figure, bounded by four lines. A square, a rectangle and a parallelogram, for instance, are quadrilaterals. In projective geometry, where sizes of angle openings and lengths of line segments are of no concern, we work with quadrilaterals of any shape and form. Every quadrilateral has two pairs of opposite sides, and every quadrilateral has two diagonals that connect opposite corners.

The initial shaded triangle forms, together with any of its three neighboring triangles, a quadrilateral. Find the neighboring triangle that shares with the initial shaded one the line that passes through the mid-point on the horizontal line. Shade the entire quadrilateral.

Through shading, organize the net of triangles into a net of quadrilaterals as far as the construction has been developed. Each immediately neighboring quadrilateral stays blank, the next one will be shaded. The result will resemble a checker board seen in perspective with shaded and non-shaded parts alternating with each other.
We now see that the three points on the horizontal line have different relations to the quadrilaterals. Two of them, the points to the left and the right, each produce lines forming opposite sides of the quadrilaterals. I designate these points as side-points $S_1$ and $S_2$. The midpoint issues diagonals, and I designate it as diagonal-point $D_1$.

For each quadrilateral, only one of its two diagonals is drawn as part of the construction. The other one is missing. As a final step, draw these diagonals. (For accuracy, utilize the points in your drawing that are most precise.)

Sufficient precision provided, you will find that the lines you now added are all parallel to each other and parallel to the initial horizontal line.

When working in a group, it is worthwhile to look at all the completed drawings and compare them. All drawings are constructed following the same principle. However, they might look markedly different. Depending on the choice of the initial triangle in its size, shape, and position, the drawings differ in perspective—they appear as if a tiled floor was looked at from a higher or a lower vantage point.

When you cover up the areas in your drawing where the quadrilaterals are most distorted, you will see the resemblance with tiled floors in paintings, for instance, by the Dutch masters Johannes Vermeer and Pieter de Hooch (see page 10). The horizontal line in the construction of a harmonic net is, in perspectival representations, the horizon line, and the three points on it are vanishing points. You will meet vanishing points and the horizon line again in chapter 6.
Lesson 2: Second drawing of a harmonic net

As in the first lesson, draw a horizontal line and choose three points on it. This time, the points are unevenly spaced.

Through each point draw a line so that the three lines form a triangle. Shade this triangle.

Proceed as in the previous lesson: Connect each corner of the shaded triangle with the point on the horizontal line it is not yet connected with. Do the same for all newly occurring points of intersection. Work methodically, step by step, and draw all lines with the greatest possible precision.

In this way, you will again create a line-woven net of triangles. You decide when to stop.

Find, for the initial shaded triangle, the neighboring triangle that shares with it the side passing through the point on the horizontal line furthest to the right. Shade this triangle.

[It is possible that in your drawing one corner of the neighboring triangle is off the page. Nevertheless, shade the portion of the triangle on the page and continue.]
Through shading, organize the net of triangles into a net of quadrilaterals. For every quadrilateral in this pattern, the point to the left as well as the point next to it must issue opposite sides. The diagonals must pass through the point to the right.

Within this pattern, so far, only one diagonal for every quadrilateral is drawn. The other diagonals are missing. Draw several of these diagonals by, again, utilizing those parts of the drawing that are precise.

If there is sufficient accuracy, you will find that the second diagonals all meet in one point on the horizontal line. This point lies between the two side-points. Designating the two diagonal-points as $D_1$ and $D_2$ and the two side-points as $S_1$ and $S_2$, the four points are, from left to right, $S_1, D_1, S_2, D_2$. While three of them were freely chosen at the beginning of the construction, the fourth point, $D_1$, was found.

In the following lesson we will ask: What determines the position of this fourth point on the horizontal line?
Lesson 3: The harmonic four points

In this lesson, we will begin differently. We will start with four lines, bounding a quadrilateral.

Draw four lines, from edge to edge of the paper. Position them so that their six points of intersection are on the paper. Shade the quadrilateral.

Four of the six points of intersection are corners (vertices) of the quadrilateral. The other two are the points in which opposite sides of the quadrilateral meet.

Draw, as a fifth line, the line through these two points of intersection.

Next, draw the two diagonals of the quadrilateral. They meet the fifth line in two more points.

We distinguish, on the fifth line, the two side-points $S_1$ and $S_2$, from the two diagonal-points $D_1$ and $D_2$. Side-points and diagonal-points alternate with each other.

These four points are called harmonic four points. There exists a significant proportional relationship among their distances from each other, expressed as the harmonic ratio. In the second volume of To the Infinite and Back Again you will hear more about this ratio.
Constructing harmonic four points

When three points on a line are randomly chosen, we can construct a fourth point such that the four are harmonic four points.

Draw a line from edge to edge of your paper. Choose three points on this line.

We distinguish the two side-points, $S_1$ and $S_2$, from the diagonal-point $D_1$. We draw from each side-point the line through the opposite corner of the triangle, constructing a quadrilateral.

We draw the missing diagonal of the quadrilateral. It meets the horizontal line in the fourth point: $D_2$. The four points, $S_1$, $D_1$, $S_2$, and $D_2$, are harmonic four points.
It is of no consequence which triangle I choose for constructing the harmonic fourth point. This I want to demonstrate in the drawings below:

All four quadrilaterals are constructed in the same relation to the freely chosen three points $S_1$, $D_1$, $S_2$. When the second diagonals are added for all quadrilaterals, we find that they all meet the horizontal line in the same point, which we designate as $D_2$.

The position of the fourth harmonic point, therefore, is not dependent on our choice of the initial triangle. It is solely dependent on the three points and their designation as side-points and diagonal-point.

When any three points on a line are given, I may choose to designate as diagonal-point any one of them. I have three choices. Each of these three choices will in turn yield a different fourth harmonic point, since diagonal-points and side-points always lie alternatingly on a line. So three random points on a line belong to three different sets of harmonic four points.
Lesson 4: A continuous movement among harmonic four points

Building on the previous lesson, we will explore a movement among harmonic four points.

We will work with a situation in which the positions of the two side-points, $S_1$ and $S_2$, on the horizontal line remain fixed. The third point, diagonal-point $D_1$, is to move from $S_2$ to $S_1$. How does the corresponding harmonic fourth point, diagonal-point $D_2$, move accordingly?

For the following figures, I chose, in the original drawings, a distance of 4 inches (approximately 10cm) between the two side-points $S_1$ and $S_2$. In seven consecutive constructions, diagonal-point $D_1$ is placed ½ inch, 1 inch, 1 ½ inches, 2 inches, 2 ½ inches, 3 inches and 3 ½ inches to the left of side-point $S_2$. I constructed the harmonic fourth point, diagonal-point $D_2$, as discussed in the previous lesson. Observe the movements of both diagonal-points.
The drawings capture a movement in seven snapshots. In thought, we can picture this movement as a continuous one. While diagonal-point $D_1$ travels on the horizontal line to the left with constant speed, from $S_2$ to $S_1$, diagonal-point $D_2$ moves to the right and then returns from the left. When diagonal-point $D_1$ is close to side-point $S_2$, diagonal-point $D_2$ is also close to $S_2$. As diagonal-point $D_1$ approaches the midpoint between the two side-points, diagonal-point $D_2$ moves to the right with increasing speed. When $D_1$ assumes the exact midpoint position, the second diagonal is parallel to the horizontal line. As soon as $D_1$ has passed through the mid-point position, we find $D_2$ on the left of $S_1$, slowing down and approaching $S_2$. While the movement of diagonal-point $D_1$ is tranquil and evenly paced, covering a finite line segment, diagonal-point $D_2$ moves and then races off to the right into the infinitely distant and returns, from the infinitely distant, on the left.

Contemplating this lawful movement among harmonic four points, we can begin to conceive the line as a whole. $D_2$ moves, from a starting point, to the right and reappears from the other side. $D_2$ moves, without changing direction, from $S_2$ to $S_1$.

When $D_1$ assumes the exact midpoint position between $S_1$ and $S_2$, projective geometry conceives the harmonic fourth point $D_2$ as the horizontal line’s point at infinity. This is, at first, an uncomfortable concept. So we will give it due attention in what follows.

*Course in Florianopolis, Brazil, in July 2016*
**Interlude**

**THE INFINITELY DISTANT POINT OF A LINE**

Let us review the exercises of the previous chapter. The first construction of a harmonic net, where all second diagonals were found to be parallel to each other and to the horizontal line, can be viewed in the context of the lessons that followed. Picture harmonic nets with $D_1$ moving from right to left, from $S_1$ to $S_2$. Diagonal-point $D_2$, in which all second diagonals meet, will move to the right with increasing speed, from $S_2$ to the infinitely distant. When $D_1$ assumes the exact mid-point position (first construction), $D_2$ cannot be an ordinary (Euclidean) point because all second diagonals are parallel to each other and to the horizontal line. Here, Euclidean geometry would come to a halt.

But in the context we are working with, where the transformation continues and diagonal-point $D_2$ reappears from the left as soon as $D_1$ has passed through the midpoint position, it becomes meaningful to relate the continuous movement of one diagonal-point ($D_1$) to a continuous movement of the other diagonal-point ($D_2$). This means that in the instance where diagonal-point $D_1$ holds the mid-point-position between the two side-points, we conceive the harmonic fourth point $D_2$ to be the (horizontal) line’s point at infinity. Before and after that instance, diagonal-point $D_2$ is a finite point on the horizontal line. It can be extremely far to the right or to the left, but that does not pose a challenge for our mental picturing.

A line has one and only one point at infinity. This point, however, cannot be pictured. The formulation, “all second diagonals meet in the horizontal line’s point at infinity when $D_1$ assumes the mid-point-position between $S_1$ and $S_2$,” is misleading as long as we connect with it the mental image of intersecting lines as we are accustomed to do. No mental image of this kind can support the concept of the point at infinity. It is, however, with full clarity of consciousness that we have pursued the path to come this far. The concept of the point at infinity does not appear arbitrary or willfully chosen. We can conceive it with necessity of thought. When we understand the principle of the harmonic net and the lawful, uninterrupted, continuous movement among harmonic four points we can see the necessity of thinking this concept quite independent of a supportive mental picture.

At this point, as a teacher, I might make a mistake. I might insist on the existence of a point at infinity and claim it to be a fundamental concept of projective geometry, which, of course, it is. But a student might honestly reply, and I have heard such comments: “I don’t believe it.” In mathematics we should not require a student to believe in something. Rather than insisting, it is more helpful to observe and acknowledge the particular challenge we all can experience with this concept. It is not easy to grasp in thought what we cannot support by any mental image because it transcends all our experiences in a finite world on which most of our mental images rest.

When our son Martin was a little boy, he overheard a lesson on projective geometry and the concept of parallels having a point at infinity in common. Afterwards he asked me, in a whisper: “Mama, where do they meet? Do they meet at Mars?” Mars, at that time in his life, was as far away as he could imagine. My honest answer, today, to his question would have to be: “It does not work that way. Parallel lines do not meet in metric (measurable) space.” Projective geometry can teach us that we actually are able to grasp in thought what we cannot find in the finite world.
After acknowledging the challenge, it will be helpful to engage in further explorations. I hope that the following chapters will make it clear that the concepts of the infinitely distant point, the infinitely distant line, and the infinitely distant plane are meaningful and not arbitrary. We will learn to see that they provide coherence and consistency that otherwise would be lacking. A whole new understanding of spatial relations and of the relation of the finite to the infinite emerges.

A one-to-one correspondence between a pencil of lines and a range of points

In the figure below, a pencil and a line that is not part of the pencil are drawn.

We choose one line of the pencil (red in the figure), and let it rotate in the point that carries the pencil. When the rotation of 180 angle degrees is completed, the position of every one of the infinitude of lines belonging to the pencil has been assumed.

We focus now on the rotation in relation to the line not belonging to the pencil. The red line meets this line in their point of intersection (circled in the figure). When the counterclockwise rotation begins, the point of intersection moves to the right and, with increasing speed, to the infinitely distant. In one instance, the rotating line is parallel to the line not belonging to the pencil. After this instance, the point of intersection reappears from the left, and—in slowing down—arrives back at the starting point when the rotation is completed.

To picture the rotation within the pencil as a continuous movement does not pose any challenge to our imagination. But we know the corresponding movement of the point of intersection through the point range to be continuous only when we conceive the point at infinity of the line. Then the point range is a wholeness, and a point can move through the entire point range. There are two orientations that the movement of the point through the entire point range can take—from left to right or from right to left—just as there are two orientations of rotation of the line within the pencil—counterclockwise or clockwise.
We can speak of a one-to-one correspondence between a pencil and a point range: For every line of the pencil there is one and only one point of the point range (the meeting point), and for every point of the point range there is one and only one line of the pencil (the line that connects the point of the point range with the point that carries the pencil). The 180 angle degree rotation within the pencil corresponds to the point movement through the entire point-range that includes the point at infinity.

*Another supportive picturing exercise*

In the figure below, I drew a horizontal and a vertical line and marked on the vertical line six points. In these six points I drew three sets of lines. Each set meets in a point on the horizontal line.

For clarity’s sake, let me call the point of intersection on the horizontal line point P (not labeled in the drawing). Now picture a transformation by point P moving on the horizontal line from left to right. The figure below depicts three stages of the transformation.

While P moves to the right, the lines that meet in P perform a partial rotation in their fixed points on the vertical line. Above the horizontal line, the rotation is counterclockwise, below the horizontal line it is clockwise. As P is moving further away and to the infinitely distant, the rotating lines become horizontal.

A course participant once shared an interesting observation about this imagination exercise: As long as you focus on the intersection P moving off to the right on the horizontal line, you are picturing lines meeting in ordinary (Euclidean) points, and you *cannot* picture those lines becoming parallel. But when, instead, you focus on the rotations of lines within the fixed points on the vertical line, you see them becoming horizontal as P moves to the infinitely distant.
The infinitude of points at infinity in space and in a plane

Every line has one and only one point at infinity. Lines that are parallel to each other have the same point at infinity. Lines that are not parallel do not have the same point at infinity but meet in an ordinary point or they are askew. (Two lines in space do not necessarily have a point in common. They have a point in common if and only if they have a plane in common. Lines in space that do not meet are called askew.)

A pencil in an ordinary (Euclidean) point represents all possible directions of lines that lie in the pencil’s plane: Every line of the plane that does not belong to the pencil is parallel to one and only one of the lines of the pencil. Thus, the lines of a pencil (in an ordinary point) represent all points at infinity of the plane. There is an infinitude of points at infinity in a plane.

Likewise, in three-dimensional space, a bundle of lines in an ordinary point represents all possible directions of lines in space. Thus, a bundle (in an ordinary point) represents all points at infinity in space.

While we cannot picture parallels having an infinitely distant point in common, we can observe and judge directions of lines. The direction of a line is an expression of the line’s point at infinity.

As we speak of a pencil or a bundle of lines in an ordinary point, we can speak of a pencil or a bundle in a point at infinity: A pencil in a point at infinity is comprised of all lines in the plane that are parallel. A bundle of lines in a point at infinity consists of all lines in space that are parallel, i.e. have the same direction.

In the figure below, two pencils are indicated with some of their lines. The parallel lines to the right belong to the pencil in the infinitely distant point of these lines.

![pencil in an ordinary point](image1.png)  ![pencil in a point at infinity](image2.png)
Chapter 2

The Theorem of Pappus

In this chapter we will work with the concept of the infinitely distant point within a geometrical context that has been known since antiquity as the theorem of Pappus. Pappus of Alexandria was a Greek mathematician whose work dates from around 320 to 340 CE. Pappus, of course, was limited in his work to Euclidean points, since the concept of the point at infinity was inconceivable at that time, a fact that is expressed in the axiom of Euclidean geometry which states that all lines in a plane intersect unless they are parallel.

In the first lesson of this chapter, the reader gets to know the theorem of Pappus in a stepwise process. In the second lesson, I will broaden Pappus’ work and include one and then two infinitely distant points. It is to be seen whether Pappus’ law is upheld and can meaningfully encompass the concept of the infinitely distant point. The concept of the line at infinity will present itself in the last construction.

Lesson 1: The theorem of Pappus

First construction

Draw two lines that are parallel to each other. On each line mark three points, all of them equidistant. As in the drawing to the right, designate the points from left to right as 1, 2, 3 on the one line and as 1’, 2’, 3’ on the other line.

Draw the lines that connect point 1 with point 2’ and point 2 with point 1’. Mark their point of intersection by circling it.

Draw the lines that connect point 1 with point 3’ and point 3 with point 1’. Mark their point of intersection.
Connect the points 2 and 3' and the points 3 and 2' and mark the meeting point of these two lines also.

You will find that the three points of intersection are collinear, i.e. they lie on a line. Draw this line and emphasize it with a colored pencil.

In this construction, the line is parallel to the two lines you started out with.

For all constructions in this chapter, as in the first construction, mark three points on each of two lines and designate them as 1, 2, 3 and 1', 2', 3' respectively.

Connect 1 with 2' and 2 with 1' and mark the meeting point of the two lines: 1 2' X 2 1'.
Connect 1 with 3' and 3 with 1' and mark the meeting point of the two lines: 1 3' X 3 1'.
Connect 2 with 3' and 3 with 2' and mark the meeting point of the two lines: 2 3' X 3 2'.

Second construction

Draw two parallel lines and choose three points on each line that are unevenly spaced (i.e. that are not equidistant). Proceed with the construction as indicated above. Is there a line of Pappus?

Third construction

Draw two lines that are not parallel to each other and choose three unevenly spaced points on each line. Is there a line of Pappus?

Fourth construction

Draw two lines that are not parallel to each other and choose three unevenly spaced points on each line. This time, scramble the order of numbering the points. [You might find that you have to adjust the location and numbering of the points in order for all three points of intersection to be on your paper.] Again, is there a line of Pappus?

[See my drawings on the next page.]
The **theorem of Pappus** of Alexandria states: When two sets of three collinear points, \(1, 2, 3\) and \(1', 2', 3'\) respectively, are given, then the three points of intersection \(1 \cdot 2' \times 2 \cdot 1', 1 \cdot 3' \times 3 \cdot 1', \) and \(2 \cdot 3' \times 3 \cdot 2'\) are collinear.

The line that carries the three points of intersection is called the **line of Pappus**, and in my drawings throughout this chapter it is highlighted in red. I will call the completed construction the **figure of Pappus**.
A hexagon hidden in the figure of Pappus

A hexagon is a polygon with six sides. Most familiar are those hexagons that are fairly regular as we find them, for instance, in the honey comb of the honey bee. However, in projective geometry, where lengths of line segments and sizes of angles are of little concern, we also work with irregular hexagons: Six points that are in the same plane are to be connected in a sequence of lines whereby each point is touched once and the starting point is also the ending point.

In all figures of Pappus we find a hexagonal structure with the six points, 1, 2, 3 and 1', 2', 3', being the corners.

In the figure at right I added arrows to the fourth construction that indicate the hexagonal structure. Starting with point 1, a line connects 1 with 2'; 2' with 3; 3 with 1'; 1' with 2; 2 with 3'; 3' with point 1. All six points are touched once in the order 1 - 2' - 3 - 1' - 2 - 3'. (Starting with the same point 1, you can, of course, transverse the hexagon in the reverse order 1 - 3' - 2 - 1' - 3 - 2' - 1.)

The corners of the hexagon alternate between the two lines.

In the case of a fairly regular hexagon we speak, in a self-evident way, of sides that are opposite each other. By numbering the six sides in their order, pairs of opposite sides are the first and the fourth side, the second and the fifth, and the third and the sixth.

In the same sense we can speak of opposite sides in every irregular hexagon. And you will find that it is the intersections of those three pairs of opposite sides in any figure of Pappus that lie on the Pappus line.

The theorem of Pappus, therefore, is also known as Pappus’ hexagon theorem and can be formulated in the following, somewhat surprising and elegant way:

**Pappus’ hexagon theorem:** If the corners of a hexagon alternatingly lie on two lines, then the three meeting points of the hexagon’s opposite sides lie on a line.
Lesson 2: Introducing infinitely distant points into the context of the theorem of Pappus

To indicate that a point P is not an ordinary (Euclidean) point but a point at infinity, it is customary to use the symbol for infinity, the figure eight $\infty$, and write $P_\infty$.

**Fifth construction**

Here, one of the six points is infinitely distant. In my construction, it is point 3.

The line through 1 and 2' intersects with the line through 2 and 1'. The intersection 1 2' X 2 1' is marked.

In order to connect point 1' with the infinitely distant point 3, you draw through 1' the line that is parallel to the line carrying point 3. The two parallel lines have point 3 in common.

The line connecting points 1 and 3' meets this parallel line. The intersection 1 3' X 3 1' is marked.
Likewise, in order to connect point 2' with the infinitely distant point 3, you draw the line through 2' that is parallel to the line carrying point 3.

The line connecting points 2 and 3' meets this parallel line in 2 3'. The point is marked.

The three points of intersection are on a line (in agreement with the theorem of Pappus)!

Enjoy finding in the above construction the hexagonal structure. [Starting at point 1, the sequence is 1 - 2' - 3 - 1' - 2 - 3' - 1. From 1, follow the line to point 2'. From 2' follow the line to the infinitely distant point 3. On the parallel line, return from the infinitely distant point 3 to point 1', from there follow the lines to point 2, to point 3', and back again to point 1. There are six corners, one of them is the infinitely distant point 3 that the three parallel lines have in common.]
**Sixth construction**

Two of the six points are infinitely distant. I am choosing points 3 and 3'.

The intersection 1 2' X 2 1' is constructed.

Constructing the intersection 1 3' X 3 1': The line through 1 that is parallel to the line carrying 3' connects 1 with the infinitely distant point 3'. Likewise, the line through 1' that is parallel to the line carrying 3 connects 1' with the infinitely distant point 3. Mark their intersection.

Similarly, construct the intersection 2 3' X 3 2'.

The three points of intersection are collinear. The law of Pappus is upheld.

It is challenging, but very rewarding, to look for the hexagonal structure in this figure of Pappus.

**Seventh construction—first encounter with the concept of the line at infinity**

The seventh construction, with again two infinitely distant points, is more difficult to understand than the two previous ones, but will lead to a first encounter with the concept of the line at infinity. For this to happen, I have chosen points 3 and 2' to be infinitely distant. Unlike in the sixth construction, we will have to connect these two points at infinity in order to complete the figure of Pappus. But how do I connect two infinitely distant points? – We will see.
The intersection 1 2' X 2 1' is shown. Since 2' is infinitely distant, the line connecting 1 with 2' must be parallel to the line carrying 2'.

The intersection 1 3' X 3 1' is shown. Since 3 is infinitely distant, the line connecting 1' with 3 must be parallel to the line carrying 3.

The line connecting 2 and 3' has been added. Where does it meet the line through 3 and 2'? How can I conceive a line connecting two infinitely distant points?
The intersections 1 2' X 2 1' and 1 3' X 3 1' determine a line. This line is drawn and emphasized in red.

It is parallel to line 2 3'!

Being parallel, the (red) line connecting the intersections 1 2' X 2 1' and 1 3' X 3 1' and the line 2 3' have the same point at infinity. This infinitely distant point is the third Pappus point, and the red line is a line of Pappus. Why?

In order to determine the intersection 2 3' X 3 2', we need to connect the two infinitely distant points 3 and 2'. Every ordinary line has one and only one infinitely distant point. Therefore, the line connecting 3 and 2'—two different points at infinity—cannot be an ordinary line. If we can speak of it at all, it must be a line that carries no Euclidean point whatsoever. It must be a line that carries only infinitely distant points. It would be aptly named “the line at infinity.”

Thus, the line 3 2' intersects line 2 3' in this line’s infinitely distant point. The intersection 2 3' X 3 2' is the infinitely distant point of line 2 3'. And line 2 3' is, as we found, parallel to the red line.

All three intersections, therefore, are on the red line. The red line is a Pappus line. In conceiving the line at infinity, the lawfulness of Pappus’ theorem is upheld.

I am aware that conceiving the line at infinity is challenging. This is the first time we encounter the challenge. In the following chapters we will continue to work with the concept of the line at infinity, and you will gradually become familiar with it.

It is possible and rewarding to look for the hexagonal structure in the seventh construction. One side of the hexagon, the line segment between the infinitely distant points 3 and 2', is a segment of the line at infinity.
**Interlude**

**A Triangle Transformation**

Picture a triangle with the corners A, B, and C. Let corner C be the apex of the imagined triangle and the side with the corners A and B its base.

Picture the lines AB, AC, and BC in their entirety.

Now—in your mind’s eye—transform the triangle: Line AB and the corners A and B may not change. Let line AC rotate in A, and line BC rotate in B, so that the apex C of the triangle moves upward and away from the base.

As the apex C moves upward, observe how the triangles change in shape. The inner angle at apex C gets smaller and smaller, the inner angles at the corners A and B get larger. At one instant, by necessity, the two rotating lines are parallel. The triangle now assumes an unfamiliar shape. Can we even speak of a triangle? But the process of transformation does not stop here. As the rotations continue, the rotating lines meet below line AB and their meeting point moves upward toward line AB.

The drawing on this page depicts four stages of the transformation. The initial triangle is triangle ABC₁. The rotations of lines AC and BC are indicated: line AC rotates counterclockwise in A, line BC clockwise in B.

The second stage is triangle ABC₂. It is also a finite triangle. The angle at its apex is recognizably smaller than the one of the first triangle.

The third stage captures the instance where the two rotating lines are parallel. They have the point at infinity C₃∞ in common. How does projective geometry know C₃∞ to be a triangle corner?

Let a point—in your mind’s eye—travel along the sides of the shaded area ABC₃∞. Every time the point changes onto a new line, we count a corner. Let us start at corner A. The point travels on line AB from A to B. In B it turns onto line B C₃∞. B is a corner. From B the point travels, on line B C₃∞, to the infinite. For the point to
return to A, it must leave line B \( C_3 \infty \) in the infinitely distant point \( C_3 \infty \) and continue on the parallel line A \( C_3 \infty \) back to A. Therefore, \( C_3 \infty \) is a triangle corner! Projective geometry knows \( ABC_3 \infty \) to be a triangle.

In the fourth stage, the rotating lines meet in \( C_4 \) below line AB. Can we speak of a triangle in this case? Let, again, a point travel along the sides of the shaded area. Starting at corner A, the point travels on line AB to B. In B, it turns onto line BC, travels through the infinite on line BC and arrives at C. In C, the point turns onto line AC, travels through the infinite on line AC and returns to the starting point A. There are three corners. \( ABC_4 \) is a triangle that stretches through the infinite. Note that the yellow shaded area is on the left or on the right of the point travelling along its edges. The shift—from left to right, from right to left, respectively—occurs in the infinitely distant, in the point at infinity of the respective line.

The triangle transformation is continuous and uninterrupted by the infinitely distant. There are shapes that we recognize as triangles that Euclidean geometry does not know as such. Our concept of “triangle” is growing.
Chapter 3

Sections of the Point Field

In this chapter we will concern ourselves with lines in a plane. They divide the plane—the plane seen as a point field—into sections. Here, Euclidean and projective geometry come to markedly different results. It will be interesting to compare them.

Let’s begin with one line in a plane and clarify what the word “section” means.

The Euclidean interpretation: (See the figure at right.) One line (line a) is given, and A and B are points that lie on either side of line a. The line connecting A and B is drawn. It intersects line a in a point that lies between points A and B. Points A and B are separated from each other by line a. They belong to two different sections. One line divides the Euclidean point field into two different sections (shaded blue and red, respectively).

For projective geometry, the situation looks different. (See the figure at right.) Projective geometry knows a line segment of line AB that Euclidean geometry does not know. It is the line segment between A and B that carries the point at infinity of line AB (orange, in the figure at right). This line segment does not intersect line a. Therefore, points A and B are not separated by line a and belong to the same section. One line does not divide the projective point field into different sections.

In regard to any number of lines, a section of the point field is comprised of all points that are not separated from each other by those lines. For any two points of a section there is a connecting line segment that does not intersect any of the lines.

Lesson 1: Three lines dividing the point field

When we think of a triangle, we normally picture a finite form with three corners and three sides. We do not concern ourselves with its surroundings, with the point field “outside” the triangle.
In projective geometry, this is different. Here, we are not asking whether a triangle is an acute, obtuse, or right triangle, whether it is equilateral or isosceles. Since the lengths of line segments and the sizes of angle openings are in constant flux in projective transformations, it would not be meaningful to pay heed to these measurable properties. Instead, in projective geometry, we are directed to look at the entire point field in regard to three lines.

First exercise

Draw three lines that are not concurrent (i.e. they must not all three meet in one point). Find the sections into which the point field is divided by these lines. By using colored pencils, shade the sections. Next, determine the shape of each section.

[My comments for assistance that you may or may not want to read before completing the exercise:]

Consider the points P and Q in the figure at right. They lie on line PQ. The line segment (highlighted in blue in the figure) from P to Q (or from Q to P) that carries the point at infinity of line PQ does not intersect any of the three lines. The points P and Q, therefore, belong to the same section (shaded blue). The finite triangle, shaded yellow, is one other section.

Let a point travel around the blue section, following the arrows as indicated in the figure at left and beginning at one corner. The point travels through the infinite two times. It turns three times. The blue section has three sides and three corners. The blue section is three-sided and triangular.]

You will find my drawing for this exercise at the top of page 41, left figure.
**Second exercise**

Transform the triangle of the first exercise as indicated in the figure at right: The corners A and B do not change. Line AC rotates counterclockwise in A, and line BC rotates clockwise in B.

Draw three stages of this transformation, as indicated below. In the second stage, the two rotating lines are parallel. In the third stage, they meet below line AB.

Find the sections of the point field and color them for each stage. It is important that your color choices for the second and third stages are consistent with your choices for the first stage. Determine also the shape of each section.

[My comments for assistance that you may or may not want to read before completing the exercise:]

The section shaded yellow in the figure at left is three-sided. One corner of this section is the point at infinity that the two parallel lines have in common.

The section shaded pink in the figure at right is three-sided. Let a point travel around the pink section by following the arrows as indicated in the figure. Count the turns (corners). One corner is the point at infinity of the two parallel lines. Here the point turns from one line onto the parallel line.]
In all three stages of this continuous triangle transformation, the point field is divided into four three-sided sections. Observe the transformation that each section undergoes.

Lesson 2: Four lines dividing the point field

First exercise

Draw four lines with all six points of intersection on your paper. (See, below, the drawing on the left.) Identify the sections of the point field by numbering them first and then shading them. Next, determine the shape of each section.

Second exercise

Transform the quadrilateral of the first exercise by changing two of the lines into a pair of parallels (see figure below, middle). Next, change the other two lines also into a pair of parallels (below, right). Through these transformations, the irregular quadrilateral ABCD in the first drawing becomes the parallelogram ABC'D' in the last drawing. Number and shade the sections in each drawing in agreement with your choice of numbers and colors in the first exercise.
Compare the three drawings and observe the changes that each section undergoes.

Comments

In all three stages of this transformation, the four lines divide the point field into seven sections. Four of the sections are three-sided, and three sections are four-sided. Referring to the figures above: The four sections (sections 2, 3, 5, and 7) that are adjacent to the finite quadrilateral (section 1) are three-sided in all stages of the transformation. The sections 1, 4, and 6 are four-sided in all stages.

You may observe, with some patience, the change that each section undergoes from the first to the third stage in the continuous transformation. Note, for instance, that two corners of sections 4 and 6 are points at infinity in the third stage.

For Euclidean geometry, the three stages do not yield the same number of sections. By not conceiving points at infinity, Euclidean geometry counts 11 sections for the first stage, 10 sections for the second stage, and only 9 sections for the third stage.

A square, a rectangle, and a parallelogram are special quadrilaterals. All quadrilaterals are bounded by four lines that meet in six points. In the case of squares, rectangles, or parallelograms, two of the six meeting points are points at infinity.

Students in grade school first meet square, rectangle, and parallelogram before they work with more irregular quadrilaterals. Projective geometry, in Waldorf Schools (Rudolf Steiner Schools), typically is taught in 11th grade. It is a beautiful experience for an 11th grader to engage in projective geometry and understand how the consistent and continuous transformation of irregular quadrilaterals transcends but also encompasses the more regular forms met in early grades.
Lesson 3: Five lines in the point field

First exercise

The five-pointed star harbors a pentagon and is created by five lines.

Draw five lines from edge to edge of your paper that form, in free-hand fashion, a five-pointed star. Identify the sections of the point field created by these five lines. Identify them by numbering them. (Don’t shade yet!)

Determine the shape of the sections. You find, besides the pentagon (which is a five-sided section), sections of different shapes. Shade the sections that are of the same shape with the same color. In order to avoid that neighboring sections of the same shape become undistinguishable from each other, emphasize their edges.

For my drawings, I used the color red for all three-sided sections, the color blue for all four-sided sections, and the color yellow for all five-sided sections. It will be easier for you to compare your work with my drawings (on page 46) if you make the same color choices.

Second exercise

On the pages following, you will find two work sheets. In each, there are five lines randomly drawn, but with all points of intersection on the paper. Use the same color choices as in the first exercise and identify and shade the sections according to their shapes.

By using block crayons you can speed up the labor of shading. Remember to identify the individual sections by numbers first, before shading them according to their shape.
Take a look at the three drawings. In all three drawings, the five-sided section (yellow) is surrounded by five three-sided sections (red). Also in all drawings, every three-sided section is surrounded by the five-sided section and two four-sided sections. Every four-sided section is surrounded by two three-sided sections and two four-sided sections.

Some sections that are stretching through the infinite in one drawing are finite in another drawing, and vice versa. How many sections stretch through the infinite in each drawing?

**Additional exercises**

1) You might want to design your own worksheet. Draw five lines from edge to edge of your paper. Any three of the lines must not lie in one point. Make sure that all (!) intersections of the five lines are on your paper. [How do I manage to have all ten points of intersection on my paper? I find the position for my ruler so that: the first two lines meet on the paper; the third line meets each of the previous two lines; the fourth line meets each of the previous three lines; and the fifth line meets each of the previous four lines.] Identify the sections of the point field by numbering them, and then color them according to their shape. As a variation of this exercise, you might want to choose two of the five lines to be parallel.

2) As a challenging exercise, imagine a transformation that morphs the first drawing on this page into the second drawing. [One of the triangular sections changes and becomes a triangle that stretches through the infinite. At the same time, one four-sided section that before stretched through the infinite, becomes finite.]

3) Even more difficult, imagine a transformation that morphs the second drawing into the third drawing. [The line between the five-sided section and the infinite three-sided section (in the second drawing) moves downward in the paper plane, through the infinite, and reappears from the other side. In consequence, the five-sided section now stretches through the infinite, as well as two of the neighboring three-sided sections. The three-sided section that stretched through the infinite in the second drawing is now finite (and tiny in my drawing). Three four-sided sections are finite.]
In the previous chapter, we worked with lines dividing the projective point field into sections. We counted the number of sections and determined their shape. Lines, of course, also divide the Euclidean point field into sections. However, Euclidean and projective geometry come to different results. Below you find a comparison.

**Projective interpretation**

1 section

2 sections

4 sections

7 sections

11 sections

**Euclidean interpretation**

1 section

2 sections

4 sections

7 sections

11 sections

16 sections
With increasing number of lines—in projective as well as in Euclidean interpretation—there are (1), 2, 4, 7, 11, 16, ... sections of the point field. Note the lawful increase among these numbers!

When we compare the numbers of sections in the two interpretations, it is striking that we need in Euclidean geometry one line less than in projective geometry for achieving the same number of sections. What does that tell us?

When we encountered, in the context of the theorem of Pappus, a construction that called for a line connecting two different points at infinity, we argued:

Every line we can draw, every line through a Euclidean point, is a line with one and only one infinitely distant point. Therefore, the line connecting two different points at infinity cannot be an ordinary line. If we can speak of it at all, it must be a line that carries no Euclidean point whatsoever. It must be a line that carries only infinitely distant points. It would be aptly named “the line at infinity.”

Euclidean geometry does not conceive points at infinity, and therefore, does not conceive the line at infinity. Projective geometry, however, by conceiving the line at infinity, interprets the results that Euclidean geometry finds, differently. It argues:

One line and the line at infinity divide the point field into 2 sections. Here are two dividing lines, not only one, at work!

Two lines and the line at infinity divide the point field into 4 sections. Here are three dividing lines at work!

Three lines and the line at infinity divide the point field into 7 sections. Here are four dividing lines at work!

Four lines and the line at infinity divide the point field into 11 sections. Here are five dividing lines at work!

Projective geometry is able to understand why Euclidean geometry seemingly needs one line less than projective geometry for achieving the same number of sections: By not conceiving the line at infinity, Euclidean geometry cannot take it into account. But the line at infinity, which it is unaware of, nevertheless functions as a dividing line. It is—for Euclidean geometry—an “invisible” boundary.
Chapter 4

The Theorem of Desargues

We can see in the work of Gérard Desargues (1591 - 1661) the earliest beginnings of the new science of projective geometry. An engineer, architect and mathematician born in Lyons, France, he was a contemporary of René Descartes (1596 - 1650). While Descartes’ mathematical work has an obvious place in today’s sciences (for instance, in the use of the so-called Cartesian coordinate-system), Desargues’ name and work is less well known.

In this chapter, I will first introduce the reader to Desargues’ two-triangle theorem. Then, as in the chapter on the theorem by Pappus, we will explore how the two-triangle theorem allows for points and lines to be infinitely distant. Consistent with Euclidean geometry, but not bound by its limitations, we will continue to conceive, and work with, points and lines at infinity.

On one hand, Desargues’ theorem shows that the first steps in the development of projective geometry were intimately connected with attempts to penetrate the visual world and to grasp its laws of perspective. When perspective was rediscovered in Renaissance painting, it gave rise to the new geometry which gradually overcame ancient boundaries of thought.

On the other hand, Desargues’ theorem allows for a great number of mental exercises and inner observations. This chapter will offer a few.

Lesson 1: Desargues’ two-triangle theorem

(See the figure at right.) When two triangles in a plane are given and their corners are designated as A, B, C and A’, B’, C’, respectively, the lines connecting the corresponding corners (the lines AA’, BB’, and CC’), in general, form a triangle (shaded bright yellow in the figure at right).

Conversely, when the corresponding sides of the triangles ABC and A’B’C’ are extended, they meet in three points of intersection (the points AB X A’B’, AC X A’C’ and BC X B’C’). In general, these three points do not lie on a line.

In the next drawing, we will construct two triangles for which the three lines that connect their corresponding corners, meet in a point.
**First construction**

Draw three lines through a point. Draw, as in the figure below, two triangles with their corresponding corners on these three lines.

Next, extend corresponding triangle sides and construct the points of intersection $AB \times A'B'$, $AC \times A'C'$ and $BC \times B'C'$. Mark these three points.

You will find that the three points of intersection are on a line! You have constructed a (two-dimensional) figure of Desargues.
**Converse construction**

Draw a line and choose three points on it. Then construct, as shown in the figures below, two triangles with their corresponding sides meeting in the three chosen points. Identify the correctly corresponding corners!
Lastly, connect the corresponding corners of the two triangles.

You will find that the three connecting lines meet in a point!

**Desargues’ two-triangle theorem:** If the three lines that connect corresponding corners of two triangles have a point in common, then the three points in which the corresponding sides intersect have a line in common. Conversely, if the three points in which the corresponding sides of two triangles intersect have a line in common, then the three lines that connect the corresponding corners have a point in common.

In the following lessons, for clarity’s sake, I will designate as point P (point of perspective) the meeting point of the three lines that connect the corresponding corners of two triangles.

Likewise, I will designate as line p (line of perspective) the line on which the three points lie that are the intersections of the corresponding sides of two triangles.

With this wording, we can express Desargues’ theorem in the following short form:

**Two triangles are perspective to a point if and only if they are perspective to a line.**

While in this chapter the two triangles ABC and A’B’C’ are placed in the same plane, the two-triangle theorem by Desargues is also true for three-dimensional space, i.e. for two triangles not belonging to the same plane. This we will explore in chapter 5.
Lesson 2: The figure of Desargues

In the first construction of the previous lesson, we began with a point of perspective P and drew two triangles that were perspective to it. Then the line of perspective p was derived. In the second construction, we began with a line of perspective p and drew two triangles that were perspective to it. Then the point of perspective P was derived. The end result of both constructions is the same kind of figure. I will call it the **figure of Desargues**.

The characteristics of a figure of Desargues are:

- There are 10 lines: Three lines pass through the point of perspective; six lines are the sides of the two triangles; the tenth line is the line of perspective.
- There are 10 points: six points are corners of the two triangles; three points are on the line of perspective; the tenth point is the point of perspective.
- On each of the ten lines there lie three of the ten points.
- Through each of the ten points there pass three of the ten lines.

**Exercise in flexible thinking**

In a figure of Desargues, any of the ten points can be interpreted as a point of perspective P and, likewise, any of the ten lines can be a line of perspective p. On the following page, the same figure of Desargues is depicted ten times, and in each one a single point is highlighted in black. This highlighted point is meant to be the point of perspective P. Find, in each case, the two triangles that are perspective to P and shade them with colored pencils. Also, find each time the line of perspective p. You may compare your solutions with my solutions, which are given on the page following.

This exercise is very rewarding since you are asked to recognize the same structure under ten different viewpoints. Keep in mind that on each of the three lines through P there lies one corner of each triangle. Note, also, that you are not allowed—and that there is no need—to add another line or point.
Lesson 3: Including infinitely distant elements in the figure of Desargues

In this lesson we will expand our work with Desargues’ two-triangle theorem by allowing for various elements to be infinitely distant.

First construction

The point of perspective P is infinitely distant. The three lines issuing from P are parallel.

Two triangles are drawn with their corners on the three parallel lines. They are perspective to P.

Corresponding sides of the triangles are extended and their intersections are marked. The three intersections are on a line (line p, yellow), in accordance with Desargues’ theorem.

Second construction

Two triangles are drawn perspective to point P so that the triangle sides AB and A’B’ are parallel to each other.

The point of intersection of these two parallel corresponding sides, point AB X A’B’, is a point at infinity.

The intersections AC X A’C’ and BC X B’C’ are constructed and connected (line p, yellow). We find that line p is parallel to the sides AB and A’B’!

Being parallels, line AB, line A’B’, and line p have the same point at infinity, i.e. the infinitely distant intersection AB X A’B’ is on line p.

The three points in which corresponding sides meet are on a line, in accordance with Desargues’ theorem. Two of the points are finite, the third is a point at infinity.
Third construction

Two triangles are drawn perspective to P so that the corner $B'$ of the second triangle is infinitely distant, i.e. the sides $A'B'$ and $C'B'$ are parallel to line $PB$.

The three points in which corresponding sides intersect are constructed and marked. Again, they are on a line (line $p$, yellow).

Fourth construction

This fourth construction is more challenging than the previous ones.

Two triangles are drawn perspective to P. The corners $B'$ and $C'$ are points at infinity, i.e. line $A'B'$ is parallel to line $PB$, and line $A'C'$ is parallel to line $PC$.

The intersections $AB \times A'B'$ and $AC \times A'C'$ are constructed, marked, and connected (line $p$, yellow). We find that the connecting line $p$ is parallel to the triangle side $BC$. This is significant!

The line connecting the two infinitely distant points $B'$ and $C'$ is the line at infinity. It intersects line $BC$ in this line's point at infinity.

Line $BC$ and line $p$ are parallel. They have the same point at infinity. The intersection $BC \times B'C'$, therefore, is on line $p$.

Again, all three points in which corresponding triangle sides intersect, are on a line. In this construction, two of the points are finite, the third is a point at infinity. Desargues’ triangle-theorem is upheld.
Fifth construction

Two triangles are drawn perspective to point P so that two pairs of corresponding triangle sides are pairs of parallels. Then the third pair of corresponding sides, by necessity, is also a pair of parallels.

The points of intersection of all three pairs of corresponding triangle side are points at infinity. They are on the line at infinity.

This line we cannot draw on the largest of papers. 😊 However, we are able to conceive it—meaningfully and consistently.

Lesson 4: Exercise in flexible thinking

We saw in lesson 2 that a figure of Desargues allows for ten different interpretations, since every one of its ten significant points can function as a point of perspective. This is also true for those figures of Desargues that include infinitely distant elements. For this exercise I have chosen a figure of Desargues that is similar to the first construction in the previous lesson: The point of perspective P is infinitely distant. The two triangles that are perspective to P, are shaded in blue, and the line of perspective p is marked (figure below, left). The same figure is shown on the right without any designations. The significant ten lines and ten points (one of them a point at infinity) are given.

On the next page, you will find this figure copied six times. In each of the otherwise identical drawings, one of the ten points is marked as point of perspective P. Find and shade the two triangles that are perspective to the respective point P. Also, find the tenth line, the line of perspective p, in each case. Remember that on each of the three lines through P there lies one corner of each triangle. Note that you must not add an additional line or point. For comparison with your results, you will find my drawings on the page following.
The infinitely large circle

Imagine a circle (in a plane) and let it grow without changing the center. Let the concentric circles get larger and, ultimately, become infinitely large.

In order to follow this growth process in thought, focus on the pencil in the center of the growing circles. Every line of this pencil is a central. (A line through the center of a circle is called a central.) Every central intersects every circle in two points, one on either side of the center. Observe, as the circles get larger, how these two points on each central move outward and further away from each other and from the center. When, ultimately, the circle is infinitely large, the two points are one. They are, for each central, the central’s infinitely distant point.

The infinitely large circle is comprised of all points at infinity of the plane.

Here, however, we cannot stop. Picturing the infinitely large circle as a circle in the conventional sense is misleading. Let us observe how, in growing circles, the circles’ curvature changes. The following exercise will support that observation.

Picture the four circles in the figure at right as four stages in a continuous growth process. All circles have one point in common, it lies on the horizontal line. The vertical line is tangent to all circles in this point. As the center moves further and further on the horizontal line to the right, the circles get larger and larger and their curvature changes. The curvature becomes increasingly more flat: The ever growing circles are approaching the vertical tangent. When the center is infinitely distant, and the circle is infinitely large, the circle is straightened.
Finally, a third imagination exercise. Consider, again, concentrically growing circles. In every point of each circle there is a tangent. The totality of a circle’s tangents is called the tangent envelope. Let us observe the tangent envelopes as the circles grow. (See the figure at right. For each circle, some tangents of the tangent envelope are indicated.)

In any given circle, there are two tangents for each central. They are perpendicular to their central. As the circles grow, the two tangents move outward, away from the center, remaining perpendicular to their central and, therefore, parallel to each other. When the circle is infinitely large, the two tangents become one line.

Summarizing our observations of these three imagination exercises, we may say: The infinitely large circle of a plane is comprised of all points at infinity of the plane. The infinitely large circle is identical with its tangent envelope. It is the line at infinity of the particular plane.

The line at infinity is a line of a very unusual character. Unlike ordinary lines, which have a distinct direction, the line at infinity is parallel to every line of the plane. It has no distinct direction or, we may say, it assumes all directions.

We cannot form a mental picture of the line at infinity, as we cannot picture a point at infinity. We need to take this realization seriously. However, the imagination exercises help us to grasp the concept. And with every new encounter with the line at infinity, with every new discovery and exercise in the course of our work, we will broaden and deepen our understanding of the line at infinity and its properties.

**Logical reasoning in forming the concept of the line at infinity**

In Euclidean geometry, a line—and no other geometric planar form or curve—has the property that it meets every other line of the same plane in one and only one point, unless the lines are parallel. In projective geometry, the restriction “unless the lines are parallel” is not upheld. In projective geometry, a line—and no other geometric planar form or curve—meets every other line of the same plane in one and only one point.

This property is true for the line at infinity: The line at infinity has one and only one point in common with every other line of the plane. The point in common is the particular line’s point at infinity.

Lines that have the same point at infinity are parallel. Since every point at infinity lies on the line at infinity, the line at infinity is parallel to every other line (of the plane). It has no distinct direction, or we may say, it assumes all directions.
The line at infinity in the theorems of Pappus and Desargues

When studying the theorems of Pappus and Desargues in the previous chapters, we encountered figures in which the concept of the line at infinity was called for. Without it, we would have to regard these instances as exceptions to the theorem.

To give an example, consider the fifth construction of lesson 3 in chapter 3. The two triangles that are perspective to a point are constructed such that the corresponding triangle sides are all pairs of parallels. Each pair, therefore, meets in a point at infinity. We can speak of these three different points at infinity being on a line only if we conceive the line at infinity. Only then is the theorem of Desargues upheld in this case. Otherwise, the fifth construction would be an exception.

With the concepts of point and line at infinity, Desargues’ theorem holds true for all triangle configurations that may occur in the theorem’s context. There are no exceptions to be considered in an uninterrupted and seamless flow of transformations.

The same is true for the theorem of Pappus. The seventh construction in chapter 2 made sense as a figure of Pappus only if we conceived the line at infinity.

In the following chapters we will again and again meet the concept of the line at infinity. We will experience more and more that the concept is not arbitrary but consistent and meaningful in the contexts we are working with.
Chapter 5

DESARGUES’ THEOREM IN THREE-DIMENSIONAL SPACE

In this chapter, we will again look at two triangles that are perspective. But unlike in chapter 4, we will look at them in three-dimensional space and develop Desargues’ theorem in its three-dimensional aspect. We will discover that the theorem in three-dimensional space is almost self-evident and provides a proof for the theorem in two dimensions which, within the framework of projective geometry, otherwise cannot be proven. The proof follows from the fact that every two-dimensional figure of Desargues can be interpreted as a projection of a three-dimensional one into a plane.

In this chapter, we will first build a line model of a three-dimensional figure of Desargues and exercise our mental faculty of geometric imagination. This will allow the reader to become intimately familiar with the principles inherent in Desargues’ figure. We will understand it as a figure of five planes intersecting each other. The lines of intersection we will represent in the model by thin wooden sticks. Building on this exercise, we will then transform three-dimensional figures of Desargues and, again, work with the concepts of points and lines at infinity. The chapter concludes, in lesson 3, with a particularly interesting transformation.

Lesson 1: Building the line model of a three-dimensional figure of Desargues

Before we actually build the line model of a three-dimensional figure of Desargues, we try to form an accurate mental picture of it. We will abstain, as long as possible, from referring to an illustration.

Look at the room you are in, at its walls, ceiling, and floor. Where any two of these surfaces meet, they meet in a straight line—unless the surfaces are not flat but curved. And in any corner of the room, three of the surfaces meet: two walls meet ceiling or floor.

How do five planes meet? Let us develop, step by step, a clear mental image of five planes interpenetrating each other. (At this point, we will not picture planes that are parallel to each other. Later, however, we will see that there is no need to exclude these instances. No three of the five planes may belong to the same sheaf of planes, and no four of the five planes may belong to the same bundle of planes.)

We begin with three planes. Two planes meet in a line. A third plane meets the two planes in two more lines. All three lines meet in a point. The three planes form a three-sided pyramid, a trihedron. The three edges of the trihedron form a tripod. The tripod’s apex lies in all three planes.

The fourth plane meets each of the three faces of the trihedron in a line, and these three lines of intersection form a triangle. The corners of the triangle lie on the lines of the tripod. Likewise, the fifth plane intersects the faces of the trihedron in three lines and forms a triangle. Its corners also lie on the lines of the tripod.
Corresponding corners of the two triangles lie on one of the three lines through the trihedron’s apex. The two triangles, therefore, are perspective to the apex.

In addition, the fourth and fifth planes also have a line in common.

We now count the lines of intersection of five planes. Three planes form a trihedron and determine three lines. The fourth and fifth planes intersect the trihedron in two triangles, adding six more lines. The tenth line is the line in which the fourth and fifth planes meet. There are ten lines in which five planes pairwise meet.

Let us look more closely at the tenth line, the line in which fourth and fifth planes meet. In each of the three faces of the trihedron a pair of corresponding triangle sides lie. Because lines that lie in one plane always meet, the corresponding triangle sides in each face meet. At the same time, one line of each pair lies in the fourth plane, and the other line of the pair in the fifth plane. This means that the meeting point of each pair must lie on the line that the fourth and the fifth planes have in common. In other words, all three meeting points of corresponding triangle sides lie on the tenth line. The two triangles are, therefore, perspective to the tenth line.

Finally, we count the significant points in the five-planes figure. One point is the apex of the trihedron which is the point of perspective for the two triangles. Six points are the corners of the two triangles. In three more points the three pairs of corresponding triangle sides meet. Thus, five planes determine ten significant points.

We have formed a mental picture of five interpenetrating planes without referring to an illustration. Those readers who cannot visualize the five planes-figure should refer to the illustration on page 68.

Please build now, with the help of a partner, a line model of a five planes-figure. The line model will represent the ten lines of intersection. You will need yarn, scissors, and ten thin wooden sticks. (I use for this purpose long skewers that are available in grocery stores.)

Here are some instructions, if you like: Begin with the edges of the trihedron and use three sticks to form a tripod. Tie the three sticks together near their upper ends. For the sake of later adjustments, do not make the knots too tight. Next, picture the fourth plane with a downward slant intersecting the trihedron and tie a triangle to the tripod, using three more sticks. Let the fifth plane intersect the tripod below the fourth plane but with an upward slant, and tie that triangle to the tripod. You have used now nine sticks and have tied seven knots. Adjust the structure so that the sticks that represent corresponding triangle sides meet. Use the tenth stick and tie all three meeting points to it. When you are done, you have tied ten knots. In every knot three sticks are tied together. On every stick there are three knots.

The next page shows photos of the stepwise process of building this structure. The three-dimensionality of a stick model is hard to capture in a photo. But you might find that, once you understand the structure, you can see the model in its three-dimensionality.
Three planes intersect in three lines. The fourth plane intersects the trihedron.

The fifth plane intersects the trihedron. Fourth and fifth plane meet in the tenth line.

You have built a line model of the three-dimensional **two-triangle theorem by Desargues**: If two triangles (in space) are perspective to a point P, then they are perspective to a line p. Conversely, if two triangles (in space) are perspective to a line p, then they are perspective to a point P.

In understanding Desargues’ figure as a five planes-figure we find the proof for Desargues’ two-triangle theorem. The proof for the first implication—if two triangles are perspective to a point P, then they are perspective to a line p—is given in our reflection on the tenth line on the previous page. The tenth line is line p while the apex of the trihedron is point P. The proof for the second implication—if two triangles are perspective to a line p, then they are perspective to a point P—might look like this:

Two triangles are perspective to a line p. This means that each pair of corresponding sides meets in a point on this line. Meeting in a point means that each pair of corresponding triangle sides lies in a plane. Three planes meet in a point. This point is the point of perspective P.

Desargues’ theorem is almost self-evident in three dimensions. Projective geometry cannot provide a proof for Desargues’ theorem in two dimensions. But every two-dimensional figure of Desargues is a
projection of a three-dimensional figure of Desargues into a plane. The proof for Desargues’ theorem in two dimensions then follows from the proof for the theorem in three dimensions.

Every three-dimensional figure of Desargues is a five planes-figure. Besides five planes, the figure has ten significant lines and ten significant points. In every one of the ten points three planes meet. In every one of the ten points three lines meet. On every one of the ten lines three points lie.

Exercises

You may use the line model you built for the following, very rewarding exercises:

1) For every one of the ten significant points of the model, find the three planes that meet in that point.

2) Every one of the ten significant points of the model can be interpreted as a point of perspective $P$. For each of the ten points, find the two triangles that are perspective to the chosen point $P$, and find the line of perspective $p$ (the tenth line).

3) Conversely, every one of the ten lines of the model can be interpreted as a line of perspective $p$. Find the triangles that are perspective to the chosen line of perspective $p$, and find the point of perspective $P$ in each case. (This exercise is, for most people, more difficult than the previous one, since we introduced the figure by starting with a point of perspective $P$.)

For the exercises you will need patience and perseverance. You will experience that some of the ten possibilities in each exercise are more challenging than others. You practice in these exercises letting go of a familiar conceptual grasp of the figure and re-interpreting it from a different angle, in agreement with the inherent principle that is expressed as Desargues’ theorem.
Lesson 2: Transforming a three-dimensional figure of Desargues

In the first four chapters of this book we worked with the geometry of the plane. We have been concerned with points and lines in a plane and formed the concepts of the point at infinity of a line and of the line at infinity of a plane. We have not looked at different planes as they intersect.

By viewing the theorem of Desargues as a theorem of five intersecting planes, we step from two-dimensional geometry into three-dimensional geometry. We are now dealing with points, lines, and planes. Now we meet, again, an idea that differs markedly between Euclidean and projective geometry. While Euclidean geometry states that two planes meet in a line unless they are parallel, projective geometry states that any two planes have a line in common. For us to be able to approach this idea, we will in this lesson transform three-dimensional figures of Desargues. The transformations will support us in forming the concept of parallel planes having their line at infinity in common.

Every two-dimensional figure of Desargues that we draw on the paper can be seen as the representation of a three-dimensional five planes-figure. The drawing at right represents such a figure: a three-sided pyramid is intersected by a fourth and a fifth plane that meet in line p. The red and the blue triangles are perspective to point P and line p.

In the following two transformations, we will alter the position of only the blue plane in the figure above. The three planes forming the three-sided pyramid and the plane of the red triangle do not change. While the blue plane is moving in a specific way, we will observe the movement of line p in which the blue and red planes meet.

**First transformation**

In this transformation, we move one corner of the blue triangle downward on its line through P. We move the corner that is, in the figure above, furthest at right. The two sides of the blue triangle that meet in this corner move accordingly. The third side of the blue triangle remains unaltered.

Picture the transformation and draw a few of the stages. Observe the three significant points on line p. Which one of these three points remains fixed? How do the other two points on line p move? How does line p move in the red plane?
The figure above shows three stages of the transformation. The starting position is an orange triangle in an orange plane, followed by a dark blue and a light blue one.

In the second stage (dark-blue), one pair of corresponding sides of the two triangles are parallels. They meet in the point at infinity that these sides have in common. And line p, in which the red and blue planes meet, is parallel to these sides. (See the figures above and at right).

Picture this three-dimensional transformation not in stages but as a continuous movement. Observe all parts of the figure as the figure transforms.
Second transformation

The second transformation begins with the figure of Desargues in the drawing at right. The line of perspective is parallel to the pair of parallel corresponding triangle sides.

Again, we transform this figure by changing only the position of the blue plane. The other four planes do not change. We lift the blue plane up so that the pair of parallel corresponding triangle sides remains a pair of parallels all the time. The third corner of the blue triangle remains fixed.

In the figure below, I depict this transformation in four stages.

![Transformation Diagram](image)

Picture this three-dimensional transformation not in stages, but as being continuous.

The blue plane rotates on an axis. The axis lies in the corner of the blue triangle that does not change. The axis is parallel to the pair of parallel corresponding triangle sides and parallel to all lines of perspective. As the blue plane rotates, the line of perspective travels, in the orange plane, away from the trihedron, through the infinite, and reappears from the other side.
When the line of perspective is infinitely distant, the blue and orange planes are parallel to each other. In this instance, the line of perspective is the line at infinity that the parallel planes have in common.

**Planes and their lines at infinity**

Let us consider planes in space. First, let us look at planes that are not parallel to each other.

Two planes that are not parallel meet in a line. This line belongs to either plane, and in either plane there are innumerable lines that are parallel to it. These parallels have the same point at infinity. This point at infinity, therefore, is a point at infinity that belongs to both planes. It lies on the line at infinity of either plane.

Now consider any other line in either of the two planes, consider lines that are not parallel to the planes’ line of intersection. For these lines there exists no parallel line in the other plane. Their point at infinity belongs to the one plane, but not to the other.

Therefore, the lines at infinity of two non-parallel planes are different lines and have one and only one point in common. Their point in common is the point at infinity of the line in which the two non-parallel planes meet.

Now let us look at parallel planes. Here it is different. For every line in the one plane there are innumerable parallel lines that lie in the other, parallel plane. Every point at infinity of the one plane is also a point at infinity of the parallel plane. The lines at infinity of parallel planes, therefore, are identical lines. Parallel planes have the same line at infinity.
Lesson 3: Triangle transformation with fixed point and line of perspective

The line of perspective in the figure at right can be seen as the carrier of a sheaf of planes. Every plane belonging to that sheaf intersects the three-sided pyramid, as the red and the blue planes do. The drawing below shows, in addition to the red and blue planes, four more planes (pink) intersecting the three-sided pyramid. (Note that the pyramid does not end with the point of perspective. Its three faces extend beyond that point.)

All six triangles are perspective to the same point of perspective P and the same line of perspective p.

In the following part of this lesson we will learn to construct triangles that are perspective to the same point of perspective P and the same line of perspective p.
Draw, as shown in the figure at right, three lines a, b, and c through a point of perspective P, and add a fourth line, the line of perspective p. Draw all lines from edge to edge of the paper.

Draw, as shown in the figure to the left, a triangle with its corners \( A_1, B_1, \) and \( C_1 \) lying on lines a, b, and c, and extend its sides to line p. The triangle sides meet line p in the points X, Y, and Z. (Adjust the drawing so that all three points X, Y, and Z are on the paper.)

Now draw a second triangle that is perspective to the same point P and the same line p.

We find that we can freely choose only one of the three corners of the second triangle. After that, the other two corners are already determined. In the figure below, I chose corner \( A_2 \) and connected it with point Z on line p. The connecting line meets line c in corner \( C_2 \). Corner \( C_2 \) I connected with point X on line p. The connecting line meets line b in corner \( B_2 \). The drawing is accurate if the third triangle side, the line connecting \( A_2 \) and \( B_2 \), passes through point Y on line p.

The figure is a familiar figure of Desargues. The two triangles are perspective to point P and line p. But it was constructed differently than the figures in chapter 4. I did not start with two triangles that are perspective to a point P, or a line p, and then find the line of perspective, or the point of perspective. Rather, one triangle and the line of perspective as well as the point of perspective were given.

Draw more triangles that are perspective to the same point P and line p. For each triangle, choose freely one corner, for instance the corner lying on line a.
In the figure above, all triangles are perspective to line p and point P. Note how the corresponding triangle sides rotate in the points X, Y, and Z on line p. The sides BC belong to the pencil in X, the sides AB to the pencil in Y, and the sides AC to the pencil in Z.

However, the drawing does not show the transformation in its entirety. It captures only a part. Each triangle corner moves across a finite line segment, not across the whole line, and the triangle sides do not perform a full rotation in their pencils. In the following exercise we want to construct a transformation of the same type, but in its entirety.

Use the worksheet on the next page. In addition to the triangle $A_1B_1C_1$ you find triangle corners $B_2, B_3, \ldots, B_9$ marked on line b. $B_{9\infty}$ is the point at infinity of line b. Side $A_1C_1$ is parallel to line p. For each of the points $B_2, B_3, \ldots, B_9$, draw the triangle that is perspective to point P and line p. For comparison, you will find my drawing on page 76.
We can now look at this transformation of triangles more closely.

The sides AC of all triangles are parallel to line p.

All triangles below line p are pointing downward; all triangles above line p, but below point P, are pointing upward; above point P, the triangles point downward again. The two families of downward pointing triangles differ: below line p, their corners A, B, C—in this order—run counterclockwise, above line p clockwise.
A special case is the triangle with corner \(B_6\). I chose \(B_6\) such that the line through \(B_6\) issuing from \(Y\) is parallel to line \(c\). At the same time—miraculously, it seems—the line connecting \(X\) and \(B_6\) is parallel to line \(a\). Both corners \(A_6\) and \(C_6\) are, therefore, points at infinity. They lie on the line at infinity which is, in this case, one of the three triangle sides. Since the line at infinity is parallel to every finite line, it is—like all the corresponding triangle sides \(AC\) in this transformation—also parallel to line \(p\).

The triangle with corner \(B_7\) represents all triangles in this transformation that have three finite corners but stretch through the infinite. The line through \(B_7\) issuing from point \(Y\) meets line \(c\) in \(C_7\) (below line \(p\)), and the line through \(B_7\) issuing from point \(X\) meets line \(a\) in \(A_7\) (also below line \(p\)). The drawing’s accuracy shows in side \(A_7C_7\) being parallel to line \(p\).

The next unusual triangle is triangle \(A_8B_8\infty C_8\). Since \(B_8\infty\) is the point at infinity of line \(b\), the lines issuing from \(X\) and \(Y\) (connecting \(X\) and \(Y\) with \(B_8\infty\)) must be parallel to line \(b\). The line issuing from \(X\) meets line \(a\) in \(A_8\), and the line issuing from \(Y\) meets line \(c\) in \(C_8\). Again, the drawing is accurate if side \(A_8C_8\) is parallel to line \(p\).

On line \(b\) there are two points for which there is no triangle to speak of. One is the intersection of line \(b\) with the line of perspective \(p\). Here, the triangle has collapsed into a line. The other position is the point of perspective \(P\). Here, the triangle has disappeared into a point.

It is interesting to explore the dynamics within the transformation. Observe, for instance, the related movements of the corners on line \(b\) and line \(c\). The corner on line \(b\) travels from \(B_5\) to \(B_7\), across a short, finite line segment, while at the same time, on line \(c\), the related corner travels almost the full length of line \(c\), from \(C_5\) through the infinite to \(C_7\).

In point \(X\), the triangle sides \(AB\) perform a full rotation. Take a ruler and follow the movement, starting from the position \(A_1B_1\). The rotation, following the sequence \(B_1, B_2 \ldots B_8, B_9, B_1\), is counter-clockwise. It is a 180 angle degree rotation as is the rotation of sides \(CB\) in \(Y\). Here, however, the rotation is clockwise.

Lastly, let us pay attention to the movement of sides \(AC\). All sides \(AC\) are parallel to line \(p\). Starting with the position \(A_1C_1\), the sides move upward with increasing speed and become, in the triangle \(A_{8\infty}B_8C_{8\infty}\), the line at infinity. Next, the sides are below line \(p\) and, moving upward with decreasing speed, eventually reappear on the paper and return to the starting position \(A_1C_1\). The sides \(AC\) perform a full rotation in the pencil of parallel lines. They traverse the entire plane.

This kind of triangle transformation, with fixed point and line of perspective, lends itself to many aspects of practice. The concepts of point and line at infinity become more familiar, not only because we learn to work with them, but rather because, in the context of a transformation like this, they present themselves as meaningful and even necessary. Without these concepts the transformation would not be continuous. We begin to experience lines and planes in their infinite wholeness.
Chapter 6

Shadows, Projections, and Linear Perspective

The theorem of Desargues, which we worked with in chapters 4 and 5, will lead us in this and the next chapter into new territory. It is remarkable how far-reaching the two-triangle theorem is. We begin to understand why this theorem is so highly regarded, and why Gérard Desargues is sometimes called the father of projective geometry—even though it took several more centuries until, in the 19th century, projective geometry emerged as an extensive body of knowledge in mathematics.

We will first develop the idea of projection. We will learn to project a form, point-by-point, into a picture plane, and to construct the shadow that an object casts. An interesting shadow triangle transformation will occur. Then we will contemplate the projection of a line and its infinitely distant point, and recognize the picture point of the point at infinity as a vanishing point. That will shed light on the principles of linear perspective that artists worked with in and since the Renaissance.

As before, I recommend that you execute the drawings that I am showing here. In addition, there will be opportunities for you to design your own drawings and explore the subject further.

Lesson 1: Projections and shadows

In the drawing at right you will recognize two triangles that are perspective to point P. Triangle ABC lies in a horizontal plane, triangle A’B’C’ in a vertical plane. According to Desargues’ theorem, the corresponding sides of the two triangles meet on line p in which horizontal and vertical planes meet. (This is not shown in the drawing).

Picture the horizontal plane to be the surface of a table. If we held a candle flame in the position of point P and if triangle A’B’C’ in the vertical plane were a triangle made from an opaque material, the triangle ABC would be the shadow that the triangle A’B’C’ casts onto the table.
On the other hand, starting with the triangle ABC in the horizontal plane, we can say: If we viewed the triangle ABC from point P, we would see in triangle A’B’C’ a triangle of the same shape, size, and positioning within our visual field that we would see if we looked directly at triangle ABC. For our view, the triangle in the vertical plane is in front of the triangle in the horizontal plane and hides it.

The two triangles, ABC and A’B’C’, are projections of each other. One we call the object, the other the picture or projection of the object. We can look at triangle ABC as being the object. Then the vertical plane is the picture plane (or plane of projection), and triangle A’B’C’ is the picture or projection of triangle ABC in the vertical plane. Or we can look at triangle A’B’C’ as being the object, and triangle ABC is the shadow (or projection) of triangle A’B’C’ in the horizontal plane.

An object can be two-dimensional or three-dimensional. A picture (or projection or shadow) is always two-dimensional.

**A method of projecting points into the vertical picture plane**

In the figure at right, the points A and B are to be projected into the vertical picture plane, with point P being the center of projection. Point A lies in the horizontal plane, point B above the horizontal plane. The heights of points P and B are drawn and give these points a definite location in relation to the horizontal as well as the vertical plane. Points H_P and H_B lie in the horizontal plane and vertically beneath point P and point B, respectively.

In order to construct the picture of point A, we draw the line that connects points P and A. The line is called line of projection or projection ray. Next, we connect points H_P and A. This line lies in the horizontal plane. It meets the line in which the horizontal plane intersects the vertical plane. In this meeting point, we draw a vertical line. It lies in the vertical picture plane and intersects the projection ray PA in A’ which is the picture (projection) of point A. (See the figure on the page following.)

Likewise, we construct the picture point of point B. We draw the projection ray PB. We connect points H_B and H_B. The connecting line lies in the horizontal plane and meets the vertical plane in a point. In this point we draw the vertical line. It intersects the projection ray PB in B’ which is the picture (projection) of point B. (See the figure on the page following.)
In the drawing below, point A’ is the picture point of A, but also of A₂ and A₃. In fact, A’ is the picture point of all points on the line of projection PA. Likewise, B’ is the picture point of B, and also of B₂, B₃, B₄, B₅ (note the position of point B₅) and all other points on the line of projection PB. While B lies above the horizontal plane, point B₂ lies in the plane. Note that point B₂ is the intersection of lines PB and H₈H₈.

Using this method, you can project any two- or three-dimensional object, point by point, into the vertical picture plane. You may practice projecting an object of your own design.

A method of drawing shadows

The construction of shadows follows the same principle explained above. In this case, a figure in the vertical plane is given, and we construct the shadow that the figure casts on the horizontal plane.

Say, a point X is given in the vertical plane. We project point X into the horizontal plane: First, we draw through P and X the line of projection PX. Next, we draw the height of point X (i.e. the vertical line in X). It meets the horizontal plane in point Hₓ. We connect points Hₓ and Hₚ and extend the line so that it meets the line of projection PX. The meeting point is the shadow point X’ in the horizontal plane.

You may draw a form in the vertical plane and construct its shadow point by point.

A beautiful example of shadows is given in the next lesson.
“The Drawing of a Lute”

The painter, draftsman, and engraver Albrecht Dürer (1471 – 1528, born and died in Nuremberg, Germany) illustrates in one of his woodcuts, “The Drawing of a Lute,” a method to achieve a perspectival image.

The object being drawn, or we might better say, the object of which the picture is being assembled point-by-point, is the lute on the table. The point from which the lute is viewed is fixed in the ring on the wall. This point we can interpret as the center of projection. The string, always taut by means of the weight fastened to the string, functions as a line of projection.

Two men are at work. One of them places the end of the string on the lute, the other measures the position where the string penetrates the plane of the open frame. That plane is the picture plane. The measured point is then marked on the white board which is attached to the frame by hinges. Point by point the image of the lute emerges. We see in the image that the lute is being viewed from above. The flute’s neck is foreshortened.

Artists, architects, and mathematicians in and after the Renaissance engaged in the study of perspective, striving to represent the world as we see it and to understand the representation. The specific vantage point of the viewer makes itself known in every image, the image changes when we view a scene from a different place. As a result of their studies, the principles and laws of linear perspective were discovered and articulated.
Lesson 2: Lowering a candle—lowering the point of projection

Picture the vertical plane in the drawing below to be an opaque surface (a piece of card board, for instance). A triangular opening is cut into it. In an otherwise dark room, behind the vertical plane, a candle (or another small light source) shines and illumines the table surface through the triangular opening. (When demonstrating the phenomenon, you will want to cover the table surface with white paper.) The candle is moved, from a position high above the table, vertically downward.

Observe the shadow outlines that appear on the table. In your demonstration, there will be a surprising, fluid and uninterrupted movement of forms.

The drawing at right captures the movement in four stages. The light source is first positioned in \( P_1 \), then it is lowered to the positions \( P_2 \), \( P_3 \), and \( P_4 \). The four positions are vertically beneath each other.

The center of projection \( P_3 \) is as high above the horizontal plane as the triangle corner \( C \) is.

In my drawing, I followed the construction principles described in the previous lesson. I projected the triangle ABC four times into the table plane, from the four different centers of projection.

Since the triangle corners A and B lie on the line in which vertical and horizontal planes meet, they are object point and picture point (point of projection) at the same time. I constructed the projection points \( C_1', C_2', C_3', \) and \( C_4' \).

Since \( P_3 \) and \( C \) are the same height above the horizontal plane, the line of projection \( P_3C \) is horizontal and, therefore, meets the table plane at infinity. The two shadow boundaries in this case are parallel to each other and to the projection line.

Note that point \( C_4' \) (the projection of \( C \) from \( P_4 \)) lies on the other side of the vertical plane. This point \( C_4' \) is needed to construct the shadow lines in this case.
The construction is a geometric treatment of a real phenomenon of illumination. As such, it allows us to understand the spatial relations between the source of light, the opaque object, and its shadow. “Projection ray” is a concept that helps to clarify these spatial relationships. It is not, however, a phenomenon of the visual world. This is important to realize because there is, unfortunately, a tendency to confuse geometrical concepts with properties of light. We often speak, for example, of “light rays” that “hit the table” or “hit the retina.” But there are no “rays of light” to be observed. “Ray” is a geometrical concept.

Moreover, real world illumination and shadow phenomena have features that are not captured by a geometric projection. For instance, a light source can be small, but it cannot be a point. It always has a visual extension. The shadow boundaries are sharp only in close proximity to the vertical plane. The brightness of the illuminated triangles on the table decreases with increasing distance from the light source. The table is indirectly illuminated by the other illuminated surfaces in the room. All these aspects modify the phenomenon. The geometrical drawing is a justifiable abstraction. But we should not forget that it does not encompass the richness of the actual, observable phenomenon. While it is worthwhile to study projection and its principles, it is important to realize what such study clarifies and what it does not clarify. Unless we look with open and unbiased eyes, the concrete and ever changing visual world—the world of images, of color, of brightness and darkness—will escape us, even though we understand quite well the spatial relationships among the things we see.

The photo below (by Craig Holdrege) shows a wonderful phenomenon of a moving shadow line. The photo was taken near The Nature Institute, looking into the opposite direction of the setting sun:

The red setting sun shines into the atmosphere and colors the sky in a pinkish hue. Below it, the blue area near the horizon, is the shadow of the earth. As the sun sinks further down below the horizon, the shadow boundary between the blue and pink hues rises. (Conversely, at dawn, looking west as the sun rises in the east, the shadow boundary moves downward.)
Lesson 3: Vanishing point and horizon line

We want to project a line that lies in the horizontal plane, into the vertical picture plane. Point P is the center of projection. The points $A_1, A_2, ... A_8$ lie on the line. (See the drawing at right.)

I constructed the picture point $A_8'$ of point $A_8$ by following the method explained in lesson 1. Since $A_1$ lies on the intersection of the vertical and the horizontal plane, it is object point and picture point at the same time. The line segment in the picture plane, from $A_1$ to $A_8'$ (blue), is the picture of the line segment in the horizontal plane, from $A_1$ to $A_8$. How do we know this to be true?

Because a line and a point (that does not lie on the line) determine a plane, point P and line $A_1A_8$ lie in the plane that intersects the vertical picture plane in line $A_1A_8'$. Every line of projection that connects P with a point on line $A_1A_8$, lies in this plane and intersects the vertical plane in a point on line $A_1A_8'$. (See the figure at right.)

In the figure, the projection rays from P to points $A_2, A_3, ..., A_7$ are added and highlighted in yellow. Each line of projection intersects the picture plane in the respective picture point. The picture points $A_2', A_3', ... , A_7'$ are marked.

We see in the drawing that the distances between the consecutive picture points $A_1', A_2', A_3', ... , A_8'$ decrease as the distances of the object points $A_1, A_2, A_3 ... A_8$ from the picture plane increase. This phenomenon—in perspective drawing—is known as foreshortening.

Now consider the entire line $A_1A_8$. Let a point, in your imagination, move on line $A_1A_8$ to the infinitely distant. Observe the movement of the respective lines of projection. They all lie in the plane determined by point P and line $A_1A_8$. The respective picture points all lie in the vertical picture plane on line $A_1A_8'$. 
When the moving point is the infinitely distant point $A_\infty$ of line $A_1A_8$, its line of projection is parallel to line $A_1A_8$. The projection ray $PA_\infty$ intersects the vertical picture plane in $A_\infty'$. $A_\infty'$ is the picture point of the point at infinity of line $A_1A_8$. In perspective drawing, this picture point is known as vanishing point.

To construct this point, we draw the projection ray $PA_\infty$, i.e. the line through $P$ that is parallel to line $A_1A_8$, and extend the line segment (blue) in the picture plane. Line $A_1A_8'$ intersects the projection ray $PA_\infty$ in $A_\infty'$. (See the drawing at right.)

The short line segment, from $A_1$ to $A_\infty'$ in the vertical plane, is the picture of the infinitely long line segment from $A_1$ to $A_\infty$ of line $A_1A_8$.

Every other line that is parallel to line $A_1A_8$—whether it lies in the horizontal table plane, or whether it lies above or below it—has the same point at infinity as line $A_1A_8$ and, therefore, the same vanishing point $A_\infty'$ in the vertical plane. The pictures of all these parallel lines are line segments that converge in the vanishing point $A_\infty'$.

If a line is horizontal, its vanishing point is as high above the horizontal table plane as the center of projection $P$. This fact is well known in the arts and in the study of linear perspective, and it is articulated in this way: The vanishing point of horizontal lines is at the observer’s eye level.

Every set of horizontal parallel lines has its own vanishing point that lies as high above the horizontal table plane as the center of projection $P$. The horizontal line on which all vanishing points of horizontal lines lie, is the picture (projection) of the line at infinity that all horizontal planes have in common. It is called the horizon or horizon line. The horizon is at the observer’s eye level.

See the figure at right. Two sets of horizontal parallel lines are drawn. The vanishing point of each set was constructed by drawing the parallel projection ray (highlighted yellow) and by constructing its intersection with the picture plane utilizing the method of lesson 1. The two vanishing points lie on the horizon line (highlighted in blue).
We can now construct the picture of the pattern on the table: The points in which the horizontal lines meet the vertical plane are object as well as picture points. We connect them with the respective vanishing point.

For any set of parallel lines, whatever direction they may have, we can determine their vanishing point in a picture plane. It is the line through the center of projection—through the vantage point—with the same direction as the set of parallels, that meets the picture plane in the vanishing point.

With this understanding in mind, we can solve almost all riddles and questions pertaining to phenomena of perspective and to perspectival representations.

The photo (by the author) shows a beautiful sunset in Western Australia. The edges of the beams of pink brightness converge in the center of the sun.
Lesson 4: Perspectival representation and vantage point

In middle school students learn to construct perspectival drawings. One step in becoming familiar with the principles of linear perspective is to construct images that represent tiled floors. Tiled floors in paintings support the impression of spatial depth.

In order to construct the image of a tiled floor, the students first draw a horizontal line on a large piece of paper and choose two points on it, one point near the middle of the line (V_C), one point near the edge of the paper (V_D). At the bottom edge of the paper they mark equidistant points. Then the students connect these equidistant points with the vanishing points V_C and V_D on the horizon line. Lastly, they draw horizontal lines through the points of intersection that occurred.

Already in the Renaissance a secret was known: Such a perspectival representation must be viewed from a certain vantage point in order to appear a true representation of a floor with square tiles. We saw all along that the position of the viewpoint (the ring in the wall in Dürer’s woodcut) plays a decisive role in the projected image. Changing the viewpoint changes the image.

So, where is the right vantage point from which the students’ picture must be viewed in order to be a convincing representation of a floor with square tiles?

In the figure at right, a floor plan is shown. The scene—floor, picture plane, and vantage point P—is depicted as seen from above. The two vanishing points V_C and V_D are shown.

The tiled floor has lines that are perpendicular to the picture plane. Lines that are perpendicular to a picture plane are called orthogonals, and their vanishing point is called the central vanishing point (V_C). The viewer must place herself with her eye being level with the horizon line and on an orthogonal that meets the picture in V_C.

Since the tiles are square, their diagonals meet the picture plane in a 45 degree angle. The line from the viewer’s eye to the vanishing point V_D must meet the picture plane also in this angle. Therefore, the right triangle PV_CV_D has two congruent angles of 45 degrees and is isosceles. The two legs of the right angle are equally long!

Therefore, if a viewer wishes to have the impression of seeing a floor made of square tiles, she must view the student’s drawing from a distance that is equal to the distance between the two vanishing points V_C and V_D! This is a secret that was already known in the Renaissance.
Construct a perspectival drawing of a tiled floor on a large piece of paper, as explained above. Then hold the drawing vertically in front of you, so that the horizon line is at your eye level and your eye in line with the orthogonal that meets the central vanishing point $V_c$. Bring the picture into a position so that its distance from your eye is about equal to the distance between the two vanishing points $V_c$ and $V_D$. Move the drawing closer to your eye and further away. Observe your impressions.

In the painting by Johannes Vermeer, *The Music Lesson*, we find a central vanishing point. (See my line analysis in the figure below. The horizon line is indicated.) The original painting (oil on canvas) measures 29 1/8 X 25 3/8 inches. The exhibition catalogue, *Johannes Vermeer*, mentions: “A pinhole with which Vermeer marked the vanishing point of the composition is visible in the paint layer.” (National Gallery of Art, Washington; Royal Cabinet of Paintings Mauritshuis, The Hague; Yale University Press, 1995, p. 128)

The central vanishing point in the composition of a painting is an element of artistic expression. Here, it is placed at the young lady's elbow. The edges of the white marbled tiles on the floor are on lines that converge in two points on the horizon line off the painting. This allows the viewer to enjoy the painting while standing at a reasonable distance from it!
Chapter 7

Homologies

A homology is a projective transformation that, again, builds on the theorem of Desargues. A homology transforms points into points, and lines into lines. A circle transforms into an ellipse, or into a parabola, or into a hyperbola. In this chapter, we will revisit the method of drawing triangles that are perspective to the same point and line of perspective, and then apply this method to transform a circle into an ellipse. The idea and definition of a homology will emerge. Next, we will work with the infinitely distant line in relation to a homology, and transform a circle into a parabola.

Lesson 1: The shadow of a circle—constructing an ellipse

Interpret the drawing at right as the rendering of a three-dimensional figure. Picture the blue triangle to be opaque and to be in a vertical plane. From behind the vertical plane, from point P, the triangle is illumined. There is a shadow space that the illumined triangle creates. The edges of the shadow space are given in the three lines through P. On all planes that intersect this shadow space the triangle casts a shadow.

We want to focus on those planes that meet the vertical plane in line p. We want to draw the shadow that the blue triangle casts on one of these planes. There are many options. By choosing one shadow point, there are no more options. The plane has been chosen. In this drawing I chose point A’ to be the shadow point of the triangle corner A.

Since a triangle and its shadow are perspective to the point of illumination (point P), the two triangles are also perspective to the line in which the object and the shadow plane meet (line p). We construct the shadow triangle according to Desargues’ theorem. (See the figure at right).
We can apply this method for constructing the shadow (the projection, or the picture) of any form. We choose a form (the object), we choose point P, and line p. Finally, we determine for one point of the object its picture point. These two points must lie on the same projection ray (i.e. on the same line through P). After that, there are no more choices.

I am choosing a circle to be the object. In the drawing at right, all choices are made. Point P, line p, and the picture point of the highest point of the circle is chosen.

A number of projection rays from P are drawn, including the two projection rays that are tangents to the circle. The picture curve will be within the range of these projection rays.

[For your own construction, some advice: For the beginning, as in my drawing at right, it is helpful to draw the object above line p and to place point P above the object. It is also helpful to determine the picture point of the (approximately) highest point of the object and to place it below line p. When you have gained some practice, you may modify any of these features.]

I will construct the picture points of all 20 points in which the chosen projection rays intersect the circle. I designate the highest of these points as 1, its picture point as point 1’. All points on the circle are numbered, clockwise, from 1 to 20. (I have labeled only some.)

Constructing the picture of point 6: The line connecting points 1 and 6 meets line p in a point that I connect with the shadow point 1’. The connecting line meets the projection ray P6 in the picture point 6’.

Likewise, I construct the picture of point 16 by connecting points 1 and 16. The line meets line p in a point that I connect with picture point 1’. The connecting line meets the projection ray P16 in picture point 16’.
Next, the pictures of all nine points on the lower half circle are constructed (see the figure at right). For greater accuracy, instead of using points 1 and 1', I construct the pictures of points 7 to 11 by using point 16 and its picture point 16' in the way I used points 1 and 1' above.

The pictures of points 12 to 15 I construct by using point 6 and its picture point 6'.

Next, the picture points of the remaining points on the upper half circle are constructed (see the figure at right).

I am using points 15 and 15' (not labeled in my drawing) to construct the pictures of points 2 to 5. I am using points 7 and 7' to construct the pictures of points 17 to 20.

The picture of the circle appears point by point. The picture of the circle is an ellipse.

The exercise allows for a great number of variations. For example, when we choose the same circle and point P and line p in the same positions, we can simply vary the position of the one picture point that we freely choose. We will construct differently shaped ellipses.
By interpreting the above construction as a shadow drawing or as a projection of the circle into another plane, we place the construction in a three-dimensional context. However, we can disregard the three-dimensional aspect of the construction and interpret it as a projective transformation within one plane. Every (object) point in the plane then corresponds to one and only one (picture) point in the same plane, and every (object) line corresponds to one and only one (picture) line in the same plane. The correspondence is called a **homology**. We can understand the above construction as a homology. It transforms the circle into a specific ellipse.

A homology is defined through a point of perspective $P$ (also called the **homology center**), a line of perspective $p$ (also called the **homology axis**), and **one pair of points in correspondence** under the homology. The two points in correspondence do not need to lie on opposite sides of line $p$. However, they must lie on the same line (projection ray) through $P$.

The point of perspective $P$ and the points on the line of perspective $p$ are **self-corresponding**, i.e. they are object point as well as picture point. The lines through $P$ (the projection rays) are self-corresponding lines, but the points on these lines (except for $P$ and the intersection with $p$) are obviously not self-corresponding. Under a homology, the properties of incidence are upheld: For instance, if an object point $A$ lies on the object line $a$, then its picture point $A'$ lies on the picture line $a'$. Also, if object lines $a$ and $b$ intersect in the object point $A$, then their picture lines $a'$ and $b'$ intersect in the picture point $A'$.

**Lesson 2: The line at infinity**

*Constructing the picture of the line at infinity (of the paper plane)*

In the drawing at right, a homology is given with the homology center $P$, the homology axis, and a pair of corresponding points, points $A$ and $A'$. We want to construct the picture of the line at infinity (of the paper plane) under this homology. We will construct it pointwise.

Every line through point $P$ has its distinct point at infinity. Let’s begin with projection ray $P_1\infty$ (yellow). Its infinitely distant point is $1_\infty$.

Following the method we practiced in lesson 1, we first connect $A$ with the object point $1_\infty$ by drawing the parallel to line $P_1\infty$ through $A$ (yellow). This line
meets the homology axis in a point that we connect with point A'. The connecting line meets the projection ray $P_{1\infty}$ in point 1'. Point 1' is the picture point of point $1_{\infty}$ under this homology.

In the same way we construct the picture points that correspond to the infinitely distant points $2_{\infty}$, $3_{\infty}$, ..., $8_{\infty}$.

We find—if our construction is fairly accurate—that all picture points 1', 2', ..., 8' are on a line. The line (blue) is parallel to line p. Under the chosen homology, the blue line is the picture line of the line at infinity of the paper plane.

**Constructing the object of the line at infinity (of the paper plane)**

Again, a homology is given with homology center P, homology axis p, and a pair of corresponding points, A and A'. We now want to construct the line of which the line at infinity (of the paper plane) is the picture. In other words, we want to construct the object line of the line at infinity.

The infinitely distant point $1_{\infty}'$ of the projection ray $P_{1\infty}'$ (yellow) is one picture point. The line through A' that is parallel to $P_{1\infty}'$ connects A' and $1_{\infty}'$, and it meets line p in a point that we connect with A. The connecting line meets the projection ray $P_{1\infty}'$ in point 1, which is the object point of point $1_{\infty}'$. (See the figure at right.)
In the same way we construct the object points of the infinitely distant points of other projection rays. (See the figure at right.)

If our construction is fairly accurate, all object points are on a line (green in the figure at right). The line is parallel to line p. The (green) line is the object line of the line at infinity under this homology; the line at infinity is its picture.

Note the difference between the two constructions: The blue line of the first construction is the line into which the homology transforms the line at infinity. It is a picture line. The green line in the second construction is the line that the homology transforms into the line at infinity. It is an object line.

Lesson 3: Circle and parabola

In the figure at right, a homology is given with homology center P, homology axis p, and a pair of corresponding points, A and A'.

Line \( \ell \) (green) is the object line of the line at infinity (of the paper plane) under this homology, i.e. the homology transforms line \( \ell \) into the line at infinity.

The object that I want to transform is, as in lesson 1, a circle. This circle, however, is special. I constructed it such that point A lies on the circle, and lines PA and \( \ell \) are tangents of the circle.

Using the same method as in lesson 1, I constructed the picture of the circle point by point. (See the page following.)
The picture of the circle—under this homology—is a parabola.

Take a closer look. Point A lies on the circle, its corresponding point A’ lies on the parabola. The projection ray PA is tangent of the circle and tangent of the parabola.

Points 10 and 16 lie on line p and are self-corresponding. Circle and parabola intersect in these points.

Since point 2 lies on line ℓ, the picture of point 2 is the infinitely distant point $2\infty'$ of the projection ray P2. The picture of line ℓ is the infinitely distant line (of the paper plane). Line ℓ is a tangent of the circle in point 2, the infinitely distant line is tangent of the parabola in point $2\infty'$.

A parabola is the projective picture of a circle under a certain homolgy. One of the tangents of a parabola is the line at infinity.

Circles, ellipses, parabolas, and hyperbolas are the conic sections. You might want to construct the picture of a circle under a homology with line ℓ intersecting the circle in two points, thus constructing a hyperbola.
Interlude

The Plane at Infinity

“Then comes the world again!”

When our two daughters were little girls, they engaged in a profound conversation that Craig, my husband, overheard, without interfering. It was a conversation in questions and answers:

“What is above the clouds?”
“The sun!”
“What is above the sun?”
The stars!”
“What is above the stars?”
The angels!”
“What is above the angels?”
“God!”
“What is above God?”
“Then comes the world again!”

The infinitely large sphere

Picture a sphere.

Picture in its center the bundle of centrals. Each central meets the sphere in two points on either side of the center.

In every point of the sphere there is a tangent plane. The tangent plane touches the sphere in that one point. The tangent plane is perpendicular to the central in that point.

Picture the sphere expanding concentrically and getting larger and larger.

Picture, in the process, the movement of the tangent planes. For each central, the two tangent planes move further and further away from the center and from each other, and are always perpendicular to their central and parallel to each other.

When the sphere grows infinitely large, the two points of the sphere on each central become one point. They become the central’s point at infinity. The two tangent planes perpendicular to their central become one plane.
The infinitely large sphere consists of all points at infinity in space.

When you continue the growth process in your imagination, the sphere reappears. We can picture it again. The two points on each central reappear from the opposite side.

The growing sphere now grows toward the center. We might be inclined to say that the sphere gets smaller again, that it contracts. However, when we consider the whole growth process as one continuous movement, we need to say that the sphere gets larger and larger, that it expands toward the center from which the growth process started.

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The infinitely large sphere is the plane at infinity

The infinitely large sphere is comprised of all points at infinity in three-dimensional space. It has the same essential characteristics that all planes have. These are:

1. The elements lying in a plane are points and lines. There is an infinitude of points and an infinitude of lines lying in a plane.
2. Every plane has with every other plane one line in common.
3. Every plane and every line have either one and only one point in common, or the line lies entirely in the plane.
4. Two (different) lines that lie in the same plane have one and only one point in common.
These characteristics are true for the infinitely large sphere:

1. All points at infinity in space lie in the infinitely large sphere. There is an infinitude of points at infinity. All lines at infinity lie in the infinitely large sphere. There is an infinitude of lines at infinity.
2. The infinitely large sphere has one line with every other plane in common. It is the particular plane’s line at infinity.
3. Every ordinary line has with the infinitely large sphere one and only one point in common—its point at infinity. Or the line is not ordinary but a line at infinity. Then it lies entirely in the infinitely large sphere.
4. Two (different) lines lying in the infinitely large sphere are two different lines at infinity. They belong to two different, non-parallel planes. Planes that are not parallel have only one point at infinity in common. Thus, two (different) lines at infinity have one, and only one, point in common.

We see that the properties that all planes have are also the properties of the infinitely large sphere. Therefore, we must regard the infinitely large sphere as a plane. It is called the plane at infinity.

The plane at infinity has one characteristic that is markedly different from all other planes: The plane at infinity is parallel to every other plane. It assumes all directions; or we might say, it has no distinct direction.
Closing

Throughout this book, we engaged in different kinds of exercises, worked with numerous geometric constructions, and reflected on the lawfulness inherent in them. With every step, we deepened and broadened our understanding of the fundamental concepts that distinguish projective geometry from Euclidean geometry.

Euclidean geometry, which assumes the limitless expansion of space in all directions, focuses on finite forms and their measurable properties, like the size of line segments, of areas, or volumes. Our notion of the number line and the Cartesian coordinate-system are a reflection of this concept of space. No positive number is a largest number, and no negative number a smallest. In endless succession, the unit length of 1 can be added on either side of the zero. There is no concept of the infinite but “endlessness,” “never ending.” There are no concepts of the line, of the plane, or of space as cohesive wholes.

By leaving measurement and quantification aside, the synthetic approach to projective geometry leads to radically different concepts of space and spatial relations. The questions asked in projective geometry are not the ones asked in Euclidean geometry. Projective geometry does not contradict Euclidean geometry, but transcends it. In our work, we experienced again and again that, in the context of a certain lawful principle, transformations of a figure are continuous and uninterrupted when we go beyond the conceptual boundaries of Euclidean geometry and conceive the unimaginable point, line, and plane at infinity. Without these concepts, transformations would come to a halt: there would be gaps and instances of form that would not be included. With these concepts, transformations are fluid, consistent, and whole.

By learning to think in transformations we form cognitive capacities. We perceive the necessity of, and learn to trust the concept of the infinitely distant. We learn to think it, although no mental picture can support it. The concept transcends all the experiences that we can have of physical things in space. It transcends all the experiences that we can have as embodied beings.

The cognitive capacities cultivated in these practices are those that we need when we turn to the phenomena of life. Life is change and transformation.

In courses at The Nature Institute, we often study projective geometry before we turn to plant or animal studies. We do not apply geometry when studying plants or animals. Rather, through geometry we enliven our thinking. As a result, many times we have gained, by the end of a week-long course, surprising and profound insights that would not have arisen, had we not also worked in projective geometry.

In Part II of “To the Infinite and Back Again,” we will continue the journey we have begun. We will study theorems that were discovered after Desargues. We will focus on the principle of “duality” or “polarity,” which is a distinguishing feature of projective geometry. We will contemplate and study the polarity of “center” and “periphery.” Part II, which is forthcoming, is meant to compliment Part I.
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While I learned projective geometry mainly from books and through my own teaching and research, I found my first teacher of projective geometry at the Hibernia School in Herne, Germany, in Peter Bütow, who was my mentor at that time.

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Bibliography


To the Infinite and Back Again provides countless exercises that foster clarity of thought and precision in imagination. This richly illustrated book is a practice-oriented introduction to projective geometry. In working through the exercises we learn to think in transformations and to experience a beautiful thought world in which ideas weave, grow, and metamorphose.

The book leads in a careful step-by-step fashion to the challenging idea of the infinite. We learn to think this mind-expanding concept, a concept that opens up whole new ways of understanding. We begin to see that everything finite gains wholeness and coherence when we conceive of the infinite.

As a fruit of the author’s many years of teaching, this workbook is intended for self-study by the lay-person and is a unique resource for high school and college math teachers.

HENRIKE HOLDREGE trained as a mathematician, a biologist, and a science teacher. In 1998, she co-founded The Nature Institute. She strives in her work to bring deeper dimensions of the world—of nature and of our inner life—to experience. She has taught in the Institute’s adult education programs since 2002, and has offered courses and workshops locally, nationally, and internationally. Her two main areas of focus are phenomenological studies of nature and mathematics as a training of thought.